Classifying Finite Simple Groups with Respect to the Number of Orbits Under the Action of the Automorphism Group

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Abstract

Let $\omega(G)$ denote the number of orbits on the (finite) group G under the action of $\mathrm{Aut}(G)$. Using the classification of finite simple groups, we prove that for any positive integer n, there is only a finite number of (non-abelian) finite simple groups G satisfying $\omega(G) \leq n$. Furthermore, we classify all finite simple groups G such that $\omega(G) \leq 17$. The latter result was obtained by computational means.

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1 Introduction

This paper continues the investigations concerning orbit numbers of finite simple groups under the action of their automorphism group which were done in [9]. There, explicit recursion formulae for $\omega(G)$ for all minimal simple groups as well as for all simple Zassenhaus groups are given. In the introduction of that paper a general description of the work done on this area can be found, as well as a list of references to related publications.

Here we treat the problem of determining all finite simple groups G for a given small value of $\omega(G)$. First of all, we reduce this problem to a finite one by showing that for any $n \in \mathbb{N}$ there are only finitely many (non-abelian) finite simple groups G satisfying $\omega(G) \leq n$ and deriving appropriate bounds, then we describe in general how to proceed algorithmically, and finally we give a concrete computational classification of all finite simple groups G such that $\omega(G) \leq 17$.

All computations were done using GAP (see [7]). The program code can be obtained from the author upon request.

1.1 Definition Throughout the whole paper, let G denote a finite group, $h(G)$ its class number and $\omega(G)$ the number of orbits on G under the action of its full automorphism group. The term 'finite simple group' should always be understood as 'non-abelian finite simple group'.

2 Bounds on orbit numbers

2.1 Theorem *(Lower bounds on $\omega(G)$ for finite simple groups G.)*

1. *For any positive integer n, there is only a finite number of finite simple groups G satisfying $\omega(G) \le n$.*
2. *If G is a finite simple group of Lie type over $\mathbb{F}_{p^f}$ of Lie rank l, then we have*
$$\omega(G) \ \ge \ \frac{h(G)}{|\mathrm{Out}(G)|} \ \ge \ \frac{p^{lf}}{6l(l+1)f} \ .$$
3. *Let $q := p^f$, p prime, $f \in \mathbb{N}$. Then we have the following bounds on $\omega(G)$ for the series of finite simple groups of Lie type:*

   (a) $\omega(\mathrm{PSL}(n,q)) \ge \dfrac{q^{n-1}}{(2-\delta_{n,2})(n,q-1)^2 f}$,

   $\omega(\mathrm{PSU}(n,q)) \ge \dfrac{q^{n-1}}{2(n,q+1)^2 f} \ \ (n \ge 3)$,

   (b) $\omega(\mathrm{O}(2n+1,q)) \ \ge \ \dfrac{q^n}{(1+\delta_{n,2}\delta_{p,2})(2,q-1)^2 f} \ \ (n \ge 2)$,

   $\omega(\mathrm{Sz}(q)) = \omega(\mathrm{PSL}(2,q)) + 2$,

   (c) $\omega(\mathrm{PSp}(2n,q)) \ \ge \ \dfrac{q^n}{(2,q-1)^2 f} \ \ (n \ge 3)$,

   (d) $\omega(\mathrm{O}^+(2n,q)) \ \ge \ \dfrac{q^n}{(4,q^n-1)(2,q-1)^2(2+4\delta_{n,4}) f} \ \ (n \ge 4)$,

   $\omega(\mathrm{O}^-(2n,q)) \ \ge \ \dfrac{q^n}{2(4,q^n+1)^2 f} \ \ (n \ge 4)$, $\ \omega({}^3\mathrm{D}_4(q)) \ \ge \ \dfrac{q^4}{3f}$,

   (e) $\omega(\mathrm{G}_2(q)) \ \ge \ \dfrac{q^2}{(1+\delta_{p,3}) f}$, $\ \omega({}^2\mathrm{G}_2(q)) \ = \ \omega(\mathrm{Ree}(q)) \ \ge \ \dfrac{q}{f}$,

   (f) $\omega(\mathrm{F}_4(q)) \ \ge \ \dfrac{q^4}{(1+\delta_{p,2}) f}$, $\ \omega({}^2\mathrm{F}_4(q)) \ = \ \omega(\mathrm{Ree}(q)) \ \ge \ \dfrac{q^2}{f}$,

   (g) $\omega(\mathrm{E}_6(q)) \ \ge \ \dfrac{q^6}{2(3,q-1)^2 f}$, $\ \omega({}^2\mathrm{E}_6(q)) \ \ge \ \dfrac{q^6}{2(3,q+1)^2 f}$,

   $\omega(\mathrm{E}_7(q)) \ \ge \ \dfrac{q^7}{(2,q-1)^2 f}$, $\ \omega(\mathrm{E}_8(q)) \ \ge \ \dfrac{q^8}{f}$.

   The exact orbit number for $\mathrm{Sz}(q)$ *is taken from [9], Theorem 3.4.*

Proof: We prove assertion (1), and get (2) 'for free'. Regarding their finite number, we do not have to take care of the sporadic simple groups, including the Tits group. We use the classification of finite simple groups (see e.g. [4], in particular p. xvi, Table 5; [8]), and prove the assertion for each of the finitely many series.

1. We consider the alternating groups. Here we are done since there is only a finite number of alternating groups whose order has less than n distinct prime divisors.

2. We consider the Chevalley groups. Let G be a simple algebraic group of Lie rank l over $\mathbb{F}_q$, where $q = p^f$. From [2], Theorem 3.7.6, we know that the simply-connected group G_{sc} related to G has exactly q^l semisimple conjugacy classes. The canonical projection of this group modulo its centre does not fuse more than $|\mathrm{Z}(G_{sc})| \leq l+1$ conjugacy classes, each (see [4], p. xvi, Table 6), hence we have

$$h(G) \ \geq \ \frac{q^l}{|\mathrm{Z}(G_{sc})|} \ \geq \ \frac{q^l}{l+1} \ .$$

Since the outer automorphism group does not fuse more than $|\mathrm{Out}(G)|$ conjugacy classes of G, each, and since considering generical isomorphisms of series of Chevalley groups we can always ensure that $|\mathrm{Out}(G)| \leq 6lf$ (see [4], p. xvi, Table 5; if this does not hold for the group G, then we can always take an isomorphic group satisfying this inequality), we get

$$\omega(G) \ \geq \ \frac{h(G)}{|\mathrm{Out}(G)|} \ \geq \ \frac{q^l}{6l(l+1)f} \ = \ \frac{p^{lf}}{6l(l+1)f} \ ,$$

hence our second assertion. Since the last expression obviously takes only for a finite number of triples (l, p, f), $l, f \in \mathbb{N}$, p prime, a value less than a given upper bound, we are done.

Inserting the actual orders of the outer automorphism groups (which we get from [4], p. xvi, Table 5) into the inequality in part (2) yields the bounds on $\omega(G)$ given in part (3). □

We can refine the bound for $\mathrm{PSL}(n, q)$:

2.2 Theorem *For $n \in \mathbb{N}$ and a prime power $q = p^f$, it holds that*

$$\omega(\mathrm{PSL}(n, q)) \geq \frac{q^{n-1}}{2(n, q-1)f} \ ,$$

hence we can omit the exponent 2 of $\gcd(n, q-1)$ *in the denominator of the bound given in Theorem 2.1.*

Proof: The q^{n-1} matrices

$$M_{(a_i)} \ := \begin{pmatrix} 0 & \cdots & 0 & (-1)^{n+1} \\ 1 & & & a_1 \\ & & 0 & a_2 \\ & \ddots & & \vdots \\ 0 & & & a_{n-2} \\ & & 1 & a_{n-1} \end{pmatrix} \in \mathrm{SL}(n, q), \ (a_1, \ldots, a_{n-1}) \in \mathbb{F}_q^{n-1}$$

have pairwisely different characteristic polynomials, hence lie in different conjugacy classes. So we have $h(\mathrm{SL}(n, q)) \geq q^{n-1}$. We use [9], Lemma 1.5. The f automorphisms induced by automorphisms of $\mathbb{F}_q$ fuse at most f of these q^{n-1} conjugacy classes, each. The automorphism $\phi : \mathrm{SL}(n, q) \to \mathrm{SL}(n, q), \ x \mapsto (x^{-1})^t$ fuses at most two of these sets, each. Finally, under the canonical projection $\pi : \mathrm{SL}(n, q) \to \mathrm{PSL}(n, q)$, at most $|\mathrm{Z}(\mathrm{SL}(n, q))| = \gcd(n, q-1)$ of these sets of conjugacy classes are identified with each other, each. Putting this together, we get the claimed bound. □

3 Simple groups by orbit number

For a computational classification of all finite simple groups with given orbit number, we firstly would like to determine all triples (l, p, f) of a prime p and positive integers l, f such that the rightmost term in Theorem 2.1, part (2) does not exceed a given upper bound $\omega_{\max}$. For this purpose, we take a look at the partial derivatives of

$$\omega(l, p, f) \ := \ \frac{p^{lf}}{6l(l+1)f}.$$

We have

$$\begin{aligned}
\frac{\partial\omega}{\partial l}\,\omega(l,p,f) &= \frac{p^{lf}(l(l+1)f\ln p - 2l - 1)}{6l^2(l+1)^2 f} \ > \ 0 \quad \text{if} \quad l \geq 3 \text{ or } p \geq 5 \text{ or } f \geq 3,\\
\frac{\partial\omega}{\partial p}\,\omega(l,p,f) &= \frac{p^{lf-1}}{6(l+1)} \ > \ 0 \quad \text{for all} \quad l, p, f,\\
\frac{\partial\omega}{\partial f}\,\omega(l,p,f) &= \frac{p^{lf}(lf\ln p - 1)}{6l(l+1)f^2} \ > \ 0 \quad \text{if} \quad l \geq 2 \text{ or } p \geq 3 \text{ or } f \geq 2,
\end{aligned}$$

and $\omega(l, p, f) < 2$ for all $2^3 = 8$ triples with $l \leq 2$, $p \leq 3$ and $f \leq 2$, hence assuming $\omega_{\max} \geq 2$ we can get our triples by a 3-dimensional version of the usual naive recursive 'contour fill' - algorithm used in computer graphics, where the borders of our area are the planes $l = 1$, $p = 2$ and $f = 1$, as well as the surface given by the equation $\omega(l, p, f) = \omega_{\max}$.

We would like to determine all finite simple groups G satisfying $\omega(G) \leq 17$ (a larger limit for $\omega(G)$ would require more elaborate methods than those used in this paper; a list of all simple groups G possibly satisfying $\omega(G) \leq 100$ together with the best bounds for – and often exact values of – $\omega(G)$ computed with the methods used here is available from the author upon request). For the groups of Lie type, using the above results we get 115 'admissible' triples (l, p, f). We check for any possible Lie type for which of these triples the right side of the respective inequality in Theorem 2.1, part (3) is not larger than 17, and there is a group of this type of Lie rank l over $\mathbb{F}_{p^f}$ which is not generically isomorphic to a group we have already considered before.
Now we can restrict our considerations to

1. The sporadic simple groups G having not more than $17 \cdot |\mathrm{Out}(G)| - 1$ conjugacy classes, these are M_{11}, M_{12}, M_{22}, M_{23}, J_1, J_2, J_3, HS, McL, He, ON, and the Tits group. The outer automorphism groups of M_{11}, M_{23}, and J_1 are trivial, hence the orbit numbers equal the class numbers: $\omega(\mathrm{M}_{11}) = 10$, $\omega(\mathrm{M}_{23}) = 17$, and $\omega(\mathrm{J}_1) = 15$. The other nine groups G have index 2 in their respective automorphism group. Since all necessary character tables are given in [4], we can determine $\omega(G)$ by reading off the cardinality of the preimage of 1 under the non-trivial character of degree 1 of $\mathrm{Aut}(G)$ from there; the results are as follows: $\omega(\mathrm{M}_{12}) = 12$, $\omega(\mathrm{M}_{22}) = 11$, $\omega(\mathrm{J}_2) = 16$, $\omega(\mathrm{J}_3) = \omega({}^2\mathrm{F}_4(2)') = 17$, $\omega(\mathrm{HS}) = 21$, $\omega(\mathrm{McL}) = 19$, $\omega(\mathrm{He}) = 26$, and $\omega(\mathrm{ON}) = 25$.

2. The alternating groups A_n for $n \leq 9$:
For $n \neq 6$, the orbits on A_n under the action of its automorphism group are in natural bijection with the partitions of n with an even number of even parts. Knowing this, we can check even by hand that $\omega(\mathrm{A}_{10}) = 22$, and see that the function $n \mapsto \omega(\mathrm{A}_n)$ is strictly growing. Also by counting partitions, we get $\omega(\mathrm{A}_5) = 4$, $\omega(\mathrm{A}_7) = 8$, $\omega(\mathrm{A}_8) = 12$ and $\omega(\mathrm{A}_9) = 16$. Since we have $\mathrm{A}_6 \cong \mathrm{PSL}(2, 9)$, we get from below that $\omega(\mathrm{A}_6) = 5$.

3. The following 77 simple groups of Lie type:

$\mathrm{PSL}(2,4)$	$\mathrm{PSL}(2,8)$	$\mathrm{PSL}(2,16)$	$\mathrm{PSL}(2,32)$	$\mathrm{PSL}(2,64)$
$\mathrm{PSL}(2,9)$	$\mathrm{PSL}(2,27)$	$\mathrm{PSL}(2,81)$	$\mathrm{PSL}(2,5)$	$\mathrm{PSL}(2,25)$
$\mathrm{PSL}(2,7)$	$\mathrm{PSL}(2,49)$	$\mathrm{PSL}(2,11)$	$\mathrm{PSL}(2,13)$	$\mathrm{PSL}(2,17)$
$\mathrm{PSL}(2,19)$	$\mathrm{PSL}(2,23)$	$\mathrm{PSL}(2,29)$	$\mathrm{PSL}(2,31)$	$\mathrm{PSL}(3,2)$
$\mathrm{PSL}(3,4)$	$\mathrm{PSL}(3,8)$	$\mathrm{PSL}(3,16)$	$\mathrm{PSL}(3,3)$	$\mathrm{PSL}(3,5)$
$\mathrm{PSL}(3,7)$	$\mathrm{PSL}(4,2)$	$\mathrm{PSL}(4,4)$	$\mathrm{PSL}(4,3)$	$\mathrm{PSL}(4,5)$
$\mathrm{PSL}(5,2)$	$\mathrm{PSL}(6,2)$			
$\mathrm{PSU}(3,4)$	$\mathrm{PSU}(3,8)$	$\mathrm{PSU}(3,32)$	$\mathrm{PSU}(3,3)$	$\mathrm{PSU}(3,5)$
$\mathrm{PSU}(3,11)$	$\mathrm{PSU}(3,17)$	$\mathrm{PSU}(4,2)$	$\mathrm{PSU}(4,4)$	$\mathrm{PSU}(4,3)$
$\mathrm{PSU}(4,5)$	$\mathrm{PSU}(4,7)$	$\mathrm{PSU}(5,2)$	$\mathrm{PSU}(5,4)$	$\mathrm{PSU}(6,2)$
$\mathrm{PSU}(9,2)$				
$\mathrm{O}(5,4)$	$\mathrm{O}(5,8)$	$\mathrm{O}(5,3)$	$\mathrm{O}(5,9)$	$\mathrm{O}(5,5)$
$\mathrm{O}(5,7)$	$\mathrm{O}(7,2)$	$\mathrm{O}(7,3)$	$\mathrm{O}(9,2)$	
$\mathrm{Sz}(8)$	$\mathrm{Sz}(32)$			
$\mathrm{PSp}(6,2)$	$\mathrm{PSp}(6,3)$	$\mathrm{PSp}(8,2)$		
$\mathrm{O}^+(8,2)$	$\mathrm{O}^+(8,3)$	$\mathrm{O}^+(8,5)$	$\mathrm{O}^+(10,2)$	$\mathrm{O}^+(10,3)$
$\mathrm{O}^-(8,2)$	$\mathrm{O}^-(8,3)$	$\mathrm{O}^-(10,2)$	$\mathrm{O}^-(10,3)$	
${}^3\mathrm{D}_4(2)$				
$\mathrm{G}_2(4)$	$\mathrm{G}_2(3)$			
$\mathrm{Ree}(27)$				
$\mathrm{F}_4(2)$				
${}^2\mathrm{E}_6(2)$				

From [9], Theorem 2.5 we know orbit number formulas for the groups $\mathrm{PSL}(2,p^f)$ in odd characteristic; in particular: $\omega(\mathrm{PSL}(2,p)) = \frac{1}{2}(p+3)$, $\omega(\mathrm{PSL}(2,p^2)) = \frac{1}{4}(p^2+2p+5)$, $\omega(\mathrm{PSL}(2,p^3)) = \frac{1}{6}(p^3+2p+9)$, and $\omega(\mathrm{PSL}(2,p^4)) = \frac{1}{8}(p^4+2p^2+4p+9)$.

In case $p = 2$, we have $\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5)$, hence $\omega(\mathrm{PSL}(2,4)) = 4$, and again from the same theorem as above we get $\omega(\mathrm{PSL}(2,8)) = \frac{8}{3} - (\frac{2}{3} - \omega(\mathrm{PSL}(2,2))) = 2+\omega(\mathrm{PSL}(2,2)) = 2+\omega(\mathrm{S}_3) = 2+h(\mathrm{S}_3) = 2+3 = 5$, and similarly, $\omega(\mathrm{PSL}(2,16)) = 7$, $\omega(\mathrm{PSL}(2,32)) = 9$ and $\omega(\mathrm{PSL}(2,64)) = 15$. A larger example for this kind of calculations is given directly after the aforementioned theorem.

From Theorem 2.1, we know that $\omega(\mathrm{Sz}(8)) = \omega(\mathrm{PSL}(2,8)) + 2 = 5 + 2 = 7$, and that $\omega(\mathrm{Sz}(32)) = \omega(\mathrm{PSL}(2,32)) + 2 = 9 + 2 = 11$.

We have $\mathrm{PSL}(3,2) \cong \mathrm{PSL}(2,7)$ and $\mathrm{PSL}(4,2) \cong \mathrm{A}_8$, so these groups already have been considered. In the proof of Theorem 2.9 in [9] it is worked out in great detail that $\omega(\mathrm{PSL}(3,3)) = 9$.

We can get exact class numbers for the groups $\mathrm{PSL}(n,q)$ and $\mathrm{PSU}(n,q)$ from [10], for the groups $\mathrm{G}_2(q)$ from [5], [3] ($h(\mathrm{G}_2(q)) = q^2 + 2q + 9$ if q is coprime to 6, and one less otherwise), and for the exceptional groups from [6]. For several further groups (namely $\mathrm{O}(5,5)$, $\mathrm{O}(5,7)$, $\mathrm{O}(7,2) \cong \mathrm{PSp}(6,2)$, $\mathrm{O}(7,3)$, $\mathrm{O}(9,2)$, $\mathrm{PSp}(6,3)$, $\mathrm{O}^-(8,2)$ and $\mathrm{O}^-(8,3)$), we can get the class number from the GAP character table library [1]. If we use for these groups G the term

$$\frac{h(G)-1}{|\mathrm{Out}(G)|} + 1 \;\leq\; \omega(G)$$

as a lower bound, then we can exclude the groups shown in Table 3.

G	$h(G)$	$\lvert\mathrm{Out}(G)\rvert$	$\omega(G) \geq$	G	$h(G)$	$\lvert\mathrm{Out}(G)\rvert$	$\omega(G) \geq$
$\mathrm{PSL}(4,4)$	84	4	21	$\mathrm{O}(7,3)$	58	2	30
$\mathrm{PSL}(6,2)$	60	2	30	$\mathrm{O}(9,2)$	81	1	81
$\mathrm{PSU}(3,17)$	106	6	18	$\mathrm{PSp}(6,3)$	74	2	38
$\mathrm{PSU}(4,4)$	94	4	24	$\mathrm{PSp}(8,2)$	81	1	81
$\mathrm{PSU}(4,5)$	97	4	25	$\mathrm{O}^+(10,2)$	97	2	49
$\mathrm{PSU}(5,2)$	47	2	24	$\mathrm{O}^-(8,2)$	39	2	20
$\mathrm{PSU}(9,2)$	402	6	67	$\mathrm{O}^-(8,3)$	112	4	29
$\mathrm{O}(5,5)$	34	2	18	$\mathrm{O}^-(10,2)$	115	2	58
$\mathrm{O}(5,7)$	52	2	26	$\mathrm{F}_4(2)$	95	2	48
$\mathrm{O}(7,2)$	30	1	30	${}^2\mathrm{E}_6(2)$	126	6	22

Table 3: The value $\frac{h(G)-1}{|\mathrm{Out}(G)|} + 1$ as a lower bound for $\omega(G)$.

For some groups, counting the orbits on the set of conjugacy classes under the action of the character table automorphism group (this is the group of all matrix automorphisms – permutations of rows and columns leaving the matrix invariant – of the matrix of irreducible characters) yields good enough bounds; in particular, we get $\omega(\mathrm{PSL}(4,3)) \geq 20$, $\omega(\mathrm{PSU}(3,11)) \geq 18$, $\omega(\mathrm{PSU}(6,2)) \geq 34$, $\omega(\mathrm{O}^+(8,2)) \geq 27$, $\omega(\mathrm{O}^+(8,3)) \geq 37$ and $\omega({}^3\mathrm{D}_4(2)) \geq 21$.

For the seven groups $\mathrm{PSU}(3,32)$, $\mathrm{PSU}(4,7)$, $\mathrm{O}(5,8)$, $\mathrm{O}(5,9)$, $\mathrm{O}^+(8,5)$, $\mathrm{O}^+(10,3)$ and $\mathrm{O}^-(10,3)$, we get our bounds by

(a) (pseudo-)randomly searching elements in the corresponding universal groups with as many different characteristic polynomials as possible,

(b) computing the number of orbits on the set of all occuring characteristic polynomials under the action of field automorphisms, and

(c) dividing this number by the product of the order of the centre of the group, the order of the group of diagonal automorphisms and the order of the group of graph automorphisms.

The resulting bounds are $\omega(\mathrm{PSU}(3,32)) \geq 18$, $\omega(\mathrm{PSU}(4,7)) \geq 23$, $\omega(\mathrm{O}(5,8)) \geq 19$, $\omega(\mathrm{O}(5,9)) \geq 21$, $\omega(\mathrm{O}^+(8,5)) \geq 24$, $\omega(\mathrm{O}^+(10,3) \geq 50$ and $\omega(\mathrm{O}^-(10,3)) \geq 28$.

For the group $\mathrm{PSU}(5,4)$ we can neither obtain good enough bounds using the methods above nor compute the orbit number by 'brute force'; but however, we can compute the conjugacy classes, their sizes and the orders of their elements, and decide this way which conjugacy classes can possibly be fused by outer automorphisms (only those of equal size and consisting of elements of the same order) and which cannot. This yields $\omega(\mathrm{PSU}(5,4)) \geq 30$.

The remaining groups G are small enough such that we can determine $\omega(G)$ by a 'brute force' - computation of the action of $\mathrm{Out}(G)$ on the set of conjugacy classes of G. The results are given in Table 4.

G	$\omega(G)$	G	$\omega(G)$	G	$\omega(G)$
$\mathrm{PSL}(3,4)$	6	$\mathrm{PSL}(5,2)$	20	$\mathrm{PSU}(4,3)$	14
$\mathrm{PSL}(3,8)$	17	$\mathrm{PSU}(3,4)$	9	$\mathrm{O}(5,4)$	12
$\mathrm{PSL}(3,16)$	20	$\mathrm{PSU}(3,8)$	10	$\mathrm{G}_2(3)$	17
$\mathrm{PSL}(3,5)$	19	$\mathrm{PSU}(3,3)$	10	$\mathrm{G}_2(4)$	24
$\mathrm{PSL}(3,7)$	16	$\mathrm{PSU}(3,5)$	10	$\mathrm{Ree}(27)$	19
$\mathrm{PSL}(4,5)$	34	$\mathrm{PSU}(4,2) \cong \mathrm{O}(5,3)$	15		

Table 4: The results of our 'brute force' computation.

Putting all this together, we get the results given in Table 5.

n	Simple groups G satisfying $\omega(G) = n$
4	$\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong \mathrm{A}_5$
5	$\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$, $\mathrm{PSL}(2,9) \cong \mathrm{A}_6$, $\mathrm{PSL}(2,8)$
6	$\mathrm{PSL}(3,4)$
7	$\mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,16)$, $\mathrm{PSL}(2,27)$, $\mathrm{Sz}(8)$
8	$\mathrm{PSL}(2,13)$, A_7
9	$\mathrm{PSL}(3,3)$, $\mathrm{PSL}(2,32)$, $\mathrm{PSU}(3,4)$
10	$\mathrm{PSL}(2,17)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSL}(2,25)$, M_{11}, $\mathrm{PSU}(3,5)$, $\mathrm{PSU}(3,8)$
11	$\mathrm{PSL}(2,19)$, M_{22}, $\mathrm{Sz}(32)$
12	$\mathrm{PSL}(4,2) \cong \mathrm{A}_8$, M_{12}, $\mathrm{O}(5,4)$
13	$\mathrm{PSL}(2,23)$
14	$\mathrm{PSU}(4,3)$
15	$\mathrm{PSU}(4,2) \cong \mathrm{O}(5,3)$, J_1, $\mathrm{PSL}(2,64)$, $\mathrm{PSL}(2,81)$
16	$\mathrm{PSL}(2,29)$, A_9, J_2, $\mathrm{PSL}(3,7)$
17	$\mathrm{PSL}(2,31)$, $\mathrm{PSL}(2,49)$, $\mathrm{G}_2(3)$, M_{23}, $\mathrm{PSL}(3,8)$, ${}^2\mathrm{F}_4(2)'$, J_3

Table 5: Simple groups G for given $\omega(G)$; if several groups are generically isomorphic, only one of them is mentioned.

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