AUTOMORPHISM GROUP ORBITS ON FINITE SIMPLE GROUPS

LEYLI JAFARI, STEFAN KOHL, AND E.A. O'BRIEN

ABSTRACT. Let $\omega(G)$ denote the number of orbits on the elements of a group G under the action of its automorphism group. We determine all finite simple groups G such that $\omega(G) \leq 100$.

1. INTRODUCTION

Let G be a finite group. Let $\omega(G)$, its *orbit number*, denote the number of orbits on the elements of G under the action of its automorphism group. In a sense, the orbit number tells us how many different 'kinds' of elements G has. Various results on orbit numbers have appeared in the literature. The study was initiated by Laffey and MacHale [14] who classified groups with orbit number at most 3 (all are solvable), showed that A_5 is the only non-solvable group with 4 orbits, and gave a structure theorem for certain solvable groups with orbit number 4. Those non-solvable groups with orbit number 5 were classified in [2]. Dantas, Garonzi and Bastos [8] classified those with orbit number 6 and showed that there are infinitely many with orbit number 7. Bastos and Dantas [3] gave structure theorems for those infinite groups which have both finite conjugacy classes and finite orbit number.

Of particular interest are the finite non-abelian simple groups. Kohl [12] determined the orbit numbers for all minimal simple groups. In [13] he showed that for every positive integer n there are only *finitely* many finite non-abelian simple groups having orbit number n and classified those with orbit number at most 17. Here we extend this classification to all simple groups G satisfying $\omega(G) \leq 100$. Like in [13], the limiting factors were the available algorithms and the available computing resources. Extending the classification significantly beyond 100 orbits will require improved

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algorithms (in the optimal case: orbit number formulas for all series of finite simple groups) or better computer hardware.

We consider the alternating groups, the sporadic simple groups, and the finite simple groups of Lie type in turn. We summarise the resulting classification in Table 2.

2. The alternating groups

The orbit number for an alternating group A_n ($n \neq 6$) equals the number of partitions of n which have an even number of even parts.

If $\omega(\mathrm{A}_n) \leq 100$ then $n \leq 15$. In summary: $\omega(\mathrm{A}_5) = 4$, $\omega(\mathrm{A}_6) = 5$, $\omega(\mathrm{A}_7) = 8$, $\omega(\mathrm{A}_8) = 12$, $\omega(\mathrm{A}_9) = 16$, $\omega(\mathrm{A}_{10}) = 22$, $\omega(\mathrm{A}_{11}) = 29$, $\omega(\mathrm{A}_{12}) = 40$, $\omega(\mathrm{A}_{13}) = 52$, $\omega(\mathrm{A}_{14}) = 69$, and $\omega(\mathrm{A}_{15}) = 90$.

3. The sporadic simple groups

The orbit number of a sporadic simple group G can be deduced from [6, 7].

If G has no outer automorphism, then its orbit number $\omega(G)$ equals its class number $h(G)$. Hence $\omega(\mathrm{M}_{11}) = 10$, $\omega(\mathrm{M}_{23}) = 17$, $\omega(\mathrm{M}_{24}) = 26$, $\omega(\mathrm{Co}_3) = 42$, $\omega(\mathrm{Co}_2) = 60$, $\omega(\mathrm{Co}_1) = 101$, $\omega(\mathrm{Fi}_{23}) = 98$, $\omega(\mathrm{Th}) = 48$, $\omega(\mathrm{J}_1) = 15$, $\omega(\mathrm{Ly}) = 53$, $\omega(\mathrm{Ru}) = 36$, $\omega(\mathrm{J}_4) = 62$, $\omega(\mathrm{B}) = 184$ and $\omega(\mathrm{M}) = 194$.

If G has index 2 in its automorphism group, then we can read off which pairs of classes are fused by outer automorphisms. In summary: $\omega(\mathrm{M}_{12}) = 12$, $\omega(\mathrm{M}_{22}) = 11$, $\omega(\mathrm{J}_2) = 16$, $\omega({}^2\mathrm{F}_4(2)') = 17$, $\omega(\mathrm{HS}) = 21$, $\omega(\mathrm{J}_3) = 17$, $\omega(\mathrm{McL}) = 19$, $\omega(\mathrm{He}) = 26$, $\omega(\mathrm{Suz}) = 37$, $\omega(\mathrm{O'N}) = 25$, $\omega(\mathrm{Fi}_{22}) = 59$, $\omega(\mathrm{HN}) = 44$, and $\omega(\mathrm{Fi}_{24}') = 97$.

4. The finite simple groups of Lie type

We record a basic observation which is surprisingly useful.

Remark 4.1. *The orbit number of a group G is at least*

$$\left\lceil 1 + \frac{h(G) - 1}{|\operatorname{Out}(G)|} \right\rceil,$$

where $h(G)$ is the number of conjugacy classes and $\operatorname{Out}(G)$ is the outer automorphism group of G.

Consider [13, Theorem 2.1, Part (2)]: if G is a simple group of Lie rank l over $\mathbb{F}_{p^f}$, then

$$\omega(G) \;\geq\; \frac{h(G)}{|\operatorname{Out}(G)|} \;\geq\; \frac{p^{lf}}{6l(l+1)f}.$$

For 312 triples (l, p, f) the rightmost expression is at most 100. For each finite simple group G determined by a triple, we check whether the lower bound for $\omega(G)$ given in [13, Theorem 2.1, Part (3)] is greater than 100. By also employing the bounds from [13, Theorem 2.2] for $\omega(\mathrm{PSL}(n, q))$, we can restrict to the following "candidate" simple groups of Lie type.

- The groups $\mathrm{PSL}(2, p^f)$
  - for $f = 1$ and primes $7 \leq p \leq 199$,
  - for $f = 2$ and primes $3 \leq p \leq 19$,
  - for $f = 3$ and $p \in \{2, 3, 5, 7\}$,
  - for $f = 4$ and $p \in \{2, 3, 5\}$,
  - for $f \in \{5, 6\}$ and $p \in \{2, 3\}$, and
  - for $f \in \{7, 8, 9\}$ and $p = 2$;

  and the groups $\mathrm{PSL}(n, q)$
  - for $n = 3$ and $q \in \{3, 4, 5, 7, 8, 9, 11, 13, 16, 19, 25\}$,
  - for $n = 4$ and $q \in \{3, 4, 5, 7, 8, 9\}$,
  - for $n \in \{5, 6\}$ and $q \in \{2, 3, 4\}$, and
  - for $n \in \{7, 8\}$ and $q = 2$.
- The groups $\mathrm{PSU}(n, q)$
  - for $n = 3$ and $q \in \{3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 23, 29, 32, 41\}$,
  - for $n = 4$ and $q \in \{2, 3, 4, 5, 7, 8, 9, 11\}$,
  - for $n = 5$ and $q \in \{2, 3, 4, 9\}$,
  - for $n = 6$ and $q \in \{2, 3, 5\}$, and
  - for $(n, q) \in \{(7, 2), (8, 2), (8, 3), (9, 2)\}$.
- The groups $\mathrm{O}(n, q)$
  - for $n = 5$ and $q \in \{4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 25, 27\}$,
  - for $n = 7$ and $q \in \{2, 3, 4, 5, 7, 9\}$, and
  - for $(n, q) \in \{(9, 2), (9, 3), (11, 2), (11, 3), (13, 2)\}$.
- The groups $\mathrm{Sz}(q)$ for $q \in \{8, 32, 128, 512\}$.
- The groups $\mathrm{PSp}(n, q)$ for $(n, q) \in \{(6, 3), (6, 5), (6, 7), (6, 9), (8, 3), (10, 3)\}$.
- The groups $\mathrm{O}^+(n, q)$
  - for $n = 8$ and $q \in \{2, 3, 4, 5, 7, 9\}$, and
  - for $(n, q) \in \{(10, 2), (10, 3), (10, 5), (12, 2), (12, 3), (14, 2)\}$.
- The groups $\mathrm{O}^-(n, q)$
  - for $n = 8$ and $q \in \{2, 3, 4, 5\}$, and
  - for $(n, q) \in \{(10, 2), (10, 3), (12, 2), (12, 3), (14, 2), (14, 3)\}$.
- The groups ${}^3\mathrm{D}_4(2)$, ${}^3\mathrm{D}_4(3)$ and ${}^3\mathrm{D}_4(4)$.
- The groups $\mathrm{G}_2(q)$ for $q \in \{3, 4, 5, 7, 8, 9, 16\}$.

- The group ${}^2\mathrm{G}_2(27) = \mathrm{Ree}(27)$.
- The groups $\mathrm{F}_4(2)$, $\mathrm{F}_4(3)$ and $\mathrm{F}_4(4)$.
- The group ${}^2\mathrm{F}_4(8) = \mathrm{Ree}(8)$.
- The groups $\mathrm{E}_6(2)$ and ${}^2\mathrm{E}_6(2)$.

We now consider each collection of groups in turn. Sometimes (lower bounds to) class or orbit numbers were known from existing sources, including [4], the Atlas [7], and Lübeck's database [10]. Explicit values were computed using GAP [11] and Magma [5]. Computations with larger groups relied on the infrastructure of [1] available in Magma, and minimal-degree permutation representations for automorphism groups of simple groups provided by Derek Holt.

Remark 4.2. In three cases the minimal-degree permutation representations were infeasibly large for direct computations. We sketch an alternative approach, in which all computations were carried out within the natural representation of the corresponding quasisimple group. Recently, De Franceschi, Liebeck and O'Brien developed algorithms to list conjugacy classes in quasisimple matrix groups G, and to decide quickly if elements of G are conjugate; see [9] for related discussion. The resulting algorithms are implemented in Magma. We used them to list explicitly the conjugacy classes for $G = \mathrm{SU}(5,9)$, $\mathrm{SU}(6,5)$, $\Omega^+(8,9)$. Let z generate $Z(G)$, the centre of G. We now readily identify the class representatives for $G/Z(G)$ as matrices in G by using our machinery to decide conjugacy between a class representative g of G and gz^i for proper divisors i of $|z|$. In all three cases, the action of the outer automorphisms on the classes of $G/Z(G)$ can be realised by action on these matrices. The field automorphism is realised by applying an appropriate Frobenius automorphism to an element of G. For $\mathrm{PSU}(5,9)$ and $\mathrm{PSU}(6,5)$, the diagonal automorphisms are realised by conjugating elements of G by an element from $\mathrm{GU}(5,9)$ and $\mathrm{GU}(6,5)$ of determinant 10 and 6 respectively. For $\mathrm{O}^+(8,9)$ we realise diagonal and graph automorphisms by conjugating elements of G by generators of $\mathrm{CGO}^+(8,9)$, the conformal group which preserves the form up to a scalar. The triality automorphism does not lift to $\Omega^+(8,9)$, but we can define a function on elements of $\Omega^+(8,9)$ which induces the automorphism on $\mathrm{O}^+(8,9)$; see [1, Section 10] for related discussion. We are grateful to Derek Holt for assistance in realising this approach.

- From [12, Theorem 2.5, Part (2)], we know formulae for the orbit numbers of $\mathrm{PSL}(2,p^f)$ in odd characteristic:
  - $\omega(\mathrm{PSL}(2,p)) = \frac{1}{2}(p+3)$,
  - $\omega(\mathrm{PSL}(2,p^2)) = \frac{1}{4}(p^2+2p+5)$,
  - $\omega(\mathrm{PSL}(2,p^3)) = \frac{1}{6}(p^3+2p+9)$,
  - $\omega(\mathrm{PSL}(2,p^4)) = \frac{1}{8}(p^4+2p^2+4p+9)$,
  - $\omega(\mathrm{PSL}(2,p^5)) = \frac{1}{10}(p^5+4p+15)$ and

– $\omega(\mathrm{PSL}(2,p^6)) = \frac{1}{12}(p^6+2p^3+2p^2+4p+15)$.

We deduce that all the candidate groups $\mathrm{PSL}(2,p^f)$, apart from $\mathrm{PSL}(2,199)$ and $\mathrm{PSL}(2,361)$, have orbit numbers at most 100.

From [12, Theorem 2.5, Part (1)], we obtain the orbit numbers for $\mathrm{PSL}(2,q)$ in characteristic 2.

The orbit numbers for $\mathrm{PSL}(3,q)$ for $q \le 8$ and $q = 16$ were computed in [13]. Since $h(\mathrm{PSL}(8,2)) = 246$ and $|\operatorname{Out}(\mathrm{PSL}(8,2))| = 2$, by Remark 4.1 we deduce that $\omega(\mathrm{PSL}(8,2)) \ge 1 + (246-1)/2 > 100$. The remaining orbit numbers for $\mathrm{PSL}(n,q)$ were computed using GAP and Magma.

- The values $\omega(\mathrm{PSU}(3,q))$ for $q \in \{3,4,5,8\}$ and $\omega(\mathrm{PSU}(4,q))$ for $q \in \{2,3\}$ were computed in [13]. Since $\mathrm{PSU}(4,8)$ has 602 conjugacy classes and $|\operatorname{Out}(\mathrm{PSL}(4,8))| = 6$, by Remark 4.1 we deduce that $\omega(\mathrm{PSU}(4,8)) \ge 102$.

  For $\mathrm{PSU}(7,2)$, $\mathrm{PSU}(8,2)$ and $\mathrm{PSU}(8,3)$, the same approach shows that there are more than 100 orbits under the action of the automorphism group.

  Note $h(\mathrm{PSU}(5,9)) = 1520$ and $|\operatorname{Out}(\mathrm{PSU}(5,9))| = 20$; and $h(\mathrm{PSU}(6,5)) = 752$ and $|\operatorname{Out}(\mathrm{PSU}(6,5))| = 12$. In each case, following Remark 4.2, we deduce that the orbit number is greater than 100.

  The orbit numbers for the remaining $\mathrm{PSU}(n,q)$ were computed using GAP and Magma.
- The value $\omega(\mathrm{O}(5,4))$ was computed in [13]. The orbit numbers for the remaining $\mathrm{O}(5,q)$, apart from $\mathrm{O}(5,17)$ and $\mathrm{O}(5,19)$, and for the remaining $\mathrm{O}(7,q)$, apart from $\mathrm{O}(7,7)$, were computed using GAP and Magma. The orbit and class number for $\mathrm{O}(9,2)$ coincide since its outer automorphism group is trivial. By Remark 4.1, we deduce that the remaining $\mathrm{O}(n,q)$ have more than 100 automorphism orbits.
- The orbit numbers for $\omega(\mathrm{Sz}(q))$ are deduced from Theorem 3.4 in [12] which states that $\omega(\mathrm{Sz}(q)) = \omega(\mathrm{PSL}(2,q)) + 2$.
- The values $\omega(\mathrm{PSp}(6,3))$ and $\omega(\mathrm{PSp}(6,5))$ were computed using GAP and Magma respectively. By Remark 4.1, we deduce that the remaining $\mathrm{PSp}(n,q)$ have more than 100 automorphism orbits.
- The values $\omega(\mathrm{O}^+(8,q))$ for $q \in \{2,3,4,5,7\}$ and $\omega(\mathrm{O}^+(10,q))$ for $q \in \{2,3\}$ were computed using GAP and Magma.

  Note $h(\mathrm{O}^+(8,9)) = 2262$ and $|\operatorname{Out}(\mathrm{O}^+(8,9))| = 48$; following Remark 4.2, we deduce that the orbit number is 348.

  By Remark 4.1, we deduce that the remaining $\mathrm{O}^+(n,q)$ have more than 100 automorphism orbits, by using sufficiently good lower bounds for the class numbers.
- The values $\omega(\mathrm{O}^-(8,q))$ for $q \in \{2,3,4\}$ and $\omega(\mathrm{O}^-(10,q))$ for $q \in \{2,3\}$ were computed using GAP and Magma. By Remark 4.1, we deduce that the remaining $\mathrm{O}^-(n,q)$ have more than 100 automorphism orbits, by using sufficiently good lower bounds for the class numbers.

- The values ${}^3\mathrm{D}_4(q)$ for $q \in \{2,3,4\}$ were computed using GAP and Magma.
- The values $\omega(\mathrm{G}_2(3))$ and $\omega(\mathrm{G}_2(4))$ were computed in [13], and the values $\omega(\mathrm{G}_2(5))$ and $\omega(\mathrm{G}_2(7))$ were computed using GAP. By Remark 4.1, we deduce that the remaining $\mathrm{G}_2(q)$ have more than 100 automorphism orbits.
- The value $\omega(\mathrm{Ree}(27))$ was computed in [13].
- The orbit number for $\mathrm{F}_4(2)$ was computed using Magma. By Remark 4.1, we deduce that $\mathrm{F}_4(3)$ and $\mathrm{F}_4(4)$ have more than 100 automorphism orbits.
- The value $\omega(\mathrm{Ree}(8))$ was determined independently by Frank Lübeck and Robert A. Wilson using the character table and insights on fusion of classes.
- The orbit number for $\mathrm{E}_6(2)$ was computed using Magma. The value $\omega({}^2\mathrm{E}_6(2))$ was determined by Wilson using an approach similar to that for $\omega(\mathrm{Ree}(8))$.

As part of this project, we determined $\omega(G)$ for some groups G omitted from our final classification because $\omega(G) > 100$; since this data may be of independent interest, we record it in Table 1.

Table 1. Some finite simple groups G with $\omega(G) > 100$.

G	$\omega(G)$	G	$\omega(G)$	G	$\omega(G)$
PSL(4, 7)	137	PSU(4, 11)	232	O(7, 9)	307
PSL(4, 8)	119	PSU(5, 9)	424	PSp(6, 5)	133
PSL(5, 4)	110	PSU(6, 3)	156	$\mathrm{O}^+(8,5)$	116
PSL(6, 3)	122	PSU(6, 5)	436	$\mathrm{O}^+(8,7)$	290
PSL(6, 4)	169	PSU(9, 2)	240	$\mathrm{O}^+(8,9)$	348
PSU(3, 23)	106	O(5, 13)	115	$\mathrm{O}^+(10,3)$	268
PSU(3, 29)	162	O(5, 25)	203	$\mathrm{O}^-(8,4)$	133
PSU(3, 41)	310	O(5, 27)	151	$\mathrm{O}^-(10,3)$	151
PSU(4, 9)	142	O(7, 5)	136	$\mathrm{E}_6(2)$	132

5. The classification

We summarise the resulting classification. We observe that there is no finite simple group G such that $\omega(G) \in \{18, 47, 49, 51, 54, 66, 68, 74, 79, 86, 94, 95, 96, 99\}$. For completeness, we include the list from [12] of those groups having orbit number at most 17; note that $\omega(\mathrm{PSL}(3,7)) = 15$, not 16 as claimed there.

Theorem 5.1. *The finite non-abelian simple groups G with $\omega(G) \leq 100$ are listed in Table 2 where each isomorphism type occurs precisely once.*

Table 2: Finite simple groups G for given $\omega(G) \leq 100$.

n	Finite simple groups G satisfying $\omega(G) = n$
4	$\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong \mathrm{A}_5$
5	$\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$, $\mathrm{PSL}(2,9) \cong \mathrm{A}_6$, $\mathrm{PSL}(2,8)$
6	$\mathrm{PSL}(3,4)$
7	$\mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,16)$, $\mathrm{PSL}(2,27)$, $\mathrm{Sz}(8)$
8	$\mathrm{PSL}(2,13)$, A_7
9	$\mathrm{PSL}(3,3)$, $\mathrm{PSL}(2,32)$, $\mathrm{PSU}(3,4)$
10	$\mathrm{PSL}(2,17)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSL}(2,25)$, M_{11}, $\mathrm{PSU}(3,5)$, $\mathrm{PSU}(3,8)$
11	$\mathrm{PSL}(2,19)$, M_{22}, $\mathrm{Sz}(32)$
12	$\mathrm{PSL}(4,2) \cong \mathrm{A}_8$, M_{12}, $\mathrm{O}(5,4)$
13	$\mathrm{PSL}(2,23)$
14	$\mathrm{PSU}(4,3)$
15	$\mathrm{PSU}(4,2) \cong \mathrm{O}(5,3)$, J_1, $\mathrm{PSL}(2,64)$, $\mathrm{PSL}(2,81)$, $\mathrm{PSL}(3,7)$
16	$\mathrm{PSL}(2,29)$, A_9, J_2
17	$\mathrm{PSL}(2,31)$, $\mathrm{PSL}(2,49)$, $\mathrm{G}_2(3)$, M_{23}, $\mathrm{PSL}(3,8)$, ${}^2\mathrm{F}_4(2)'$, J_3
18	
19	$\mathrm{PSL}(3,5)$, McL, $\mathrm{Ree}(27)$
20	$\mathrm{PSL}(2,37)$, $\mathrm{PSL}(5,2)$, $\mathrm{PSL}(3,16)$
21	$\mathrm{PSL}(2,128)$, $\mathrm{PSL}(4,3)$, HS, ${}^3\mathrm{D}_4(2)$, $\mathrm{O}(5,8)$
22	$\mathrm{PSL}(2,41)$, A_{10}
23	$\mathrm{PSL}(2,43)$, $\mathrm{Sz}(128)$
24	$\mathrm{PSL}(2,125)$, $\mathrm{G}_2(4)$
25	$\mathrm{PSL}(2,47)$, $\mathrm{O'N}$
26	M_{24}, He
27	$\mathrm{O}(5,5)$, $\mathrm{PSL}(2,243)$, $\mathrm{O}^+(8,2)$
28	$\mathrm{PSL}(2,53)$
29	A_{11}, $\mathrm{PSU}(3,9)$
30	$\mathrm{O}(7,2)$, $\mathrm{PSU}(5,2)$, $\mathrm{PSU}(3,11)$
31	$\mathrm{PSL}(2,59)$
32	$\mathrm{PSL}(2,61)$, $\mathrm{PSL}(3,9)$
33	$\mathrm{O}^-(8,2)$
34	$\mathrm{PSU}(3,7)$, $\mathrm{PSL}(4,5)$, $\mathrm{PSU}(5,4)$, $\mathrm{PSU}(6,2)$
35	$\mathrm{PSL}(2,67)$, $\mathrm{PSU}(4,4)$
36	$\mathrm{PSL}(4,4)$, Ru
37	$\mathrm{PSL}(2,71)$, $\mathrm{PSL}(2,121)$, $\mathrm{PSL}(2,256)$, Suz
38	$\mathrm{PSL}(2,73)$, $\mathrm{O}^+(8,3)$
39	$\mathrm{PSL}(3,13)$
40	A_{12}, $\mathrm{PSU}(3,16)$
	To be continued.

Continued.	
n	Simple groups G satisfying $\omega(G) = n$
41	PSL(2, 79), O(5, 9)
42	PSU(3, 32), Co_3
43	PSL(2, 83), O(5, 7)
44	$\mathrm{G}_2(5)$, PSL(6, 2), HN
45	O(5, 16)
46	PSL(2, 89)
47	
48	Th
49	
50	PSL(2, 97), PSL(2, 169), PSp(6, 3)
51	
52	PSL(2, 101), A_{13}, O(7, 3)
53	PSL(2, 103), Ly
54	
55	PSL(2, 107)
56	PSL(2, 109), ${}^3\mathrm{D}_4(3)$
57	Ree(8)
58	PSL(2, 113)
59	Fi_{22}
60	Co_2
61	PSL(2, 343), PSL(2, 512)
62	PSU(3, 17), $\mathrm{F}_4(2)$, J_4
63	Sz(512)
64	PSU(4, 5)
65	PSL(2, 127)
66	
67	PSL(2, 131)
68	
69	PSL(2, 729), A_{14}
70	PSL(2, 137)
71	PSL(2, 139)
72	PSL(3, 25), PSL(5, 3), $\mathrm{G}_2(7)$
73	PSL(3, 11)
74	
75	PSL(3, 19), O(7, 4)
76	PSL(2, 149), PSU(4, 7)
77	PSL(2, 151), $\mathrm{O}^-(8, 3)$, PSL(7, 2)
78	${}^3\mathrm{D}_4(4)$
	To be continued.

Continued.	
n	Simple groups G satisfying $\omega(G) = n$
79	
80	$\mathrm{PSL}(2, 157)$
81	$\mathrm{O}(9, 2)$
82	$\mathrm{PSL}(2, 289)$
83	$\mathrm{PSL}(2, 163)$
84	$\mathrm{O}^+(10, 2)$, $\mathrm{O}^+(8, 4)$
85	$\mathrm{PSL}(2, 167)$, $\mathrm{PSL}(4, 9)$
86	
87	$\mathrm{O}(5, 11)$
88	$\mathrm{PSL}(2, 173)$, $\mathrm{PSL}(2, 625)$
89	$\mathrm{PSU}(5, 3)$
90	A_{15}
91	$\mathrm{PSL}(2, 179)$, ${}^2\mathrm{E}_6(2)$
92	$\mathrm{PSL}(2, 181)$
93	$\mathrm{O}^-(10, 2)$
94	
95	
96	
97	$\mathrm{PSL}(2, 191)$, Fi'_{24}
98	$\mathrm{PSL}(2, 193)$, Fi_{23}
99	
100	$\mathrm{PSL}(2, 197)$, $\mathrm{PSU}(3, 13)$

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Department of Mathematics, University of Kurdistan, P.O. Box: 6, Sanandaj, Iran

E-mail address: l.jafari@sci.uok.ac.ir

Hessestrasse 18, 71263 Weil der Stadt, Germany

E-mail address: sk239@st-andrews.ac.uk

Department of Mathematics, University of Auckland, Auckland, New Zealand

E-mail address: e.obrien@auckland.ac.nz