THE COLLATZ CONJECTURE IN A GROUP THEORETIC CONTEXT

STEFAN KOHL

ABSTRACT. In this paper we exhibit a permutation group which acts transitively on $\mathbb{N}_0$ if and only if the Collatz conjecture holds. We also give an infinite series of finitely generated simple groups many of which contain this group as a subgroup, and whose intersection is isomorphic to Thompson’s group $V$.

1. INTRODUCTION

By $r(m)$ we denote the residue class $r + m\mathbb{Z}$, where we assume that $0 \leq r < m$. The Collatz conjecture asserts that iterated application of the mapping

$$C : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \in 0(2), \\ 3n + 1 & \text{if } n \in 1(2), \end{cases}$$

to any positive integer yields 1 after a finite number of steps (cf. Lagarias [7], [8]).

The mapping $C$ is surjective, but not injective. It is affine on residue classes, and it maps negative to negative and nonnegative to nonnegative integers. The most basic bijective mappings which share the latter properties are those which interchange two disjoint residue classes:

**Definition 1.1.** Given disjoint residue classes $r_1(m_1)$ and $r_2(m_2)$ of $\mathbb{Z}$, let the class transposition $\tau_{r_1(m_1), r_2(m_2)}$ be the permutation which interchanges $r_1 + km_1$ and $r_2 + km_2$ for each integer $k$ and which fixes all other points.

Note that the set of all class transpositions generates a countable simple group $CT(\mathbb{Z}) \leq Sym(\mathbb{Z})$ which has a rich class of subgroups, cf. [5]. In this paper we exhibit subgroups of $CT(\mathbb{Z})$ which act transitively on the set of nonnegative integers in their support if and only if the Collatz conjecture holds:

**Proposition 1.2.** The following hold:

a) The group $G_C := \langle \tau_{1(2), 4(6)}, \tau_{1(3), 2(6)}, \tau_{2(3), 4(6)} \rangle$ acts transitively on $\mathbb{N} \setminus 0(6)$ if and only if the Collatz conjecture holds.

b) The group $G_T := \langle \tau_{0(2), 1(2)}, \tau_{1(2), 2(4)}, \tau_{1(4), 2(6)} \rangle$ acts transitively on $\mathbb{N}_0$ if and only if the Collatz conjecture holds.

By Corollary 3.7 in [5], the following subgroups of $CT(\mathbb{Z})$ are simple as long as $2 \in \mathbb{P}$:

**Definition 1.3.** Given a set $\mathbb{P}$ of prime numbers, let $CT_p(\mathbb{Z}) \leq CT(\mathbb{Z})$ denote the subgroup which is generated by all class transpositions $\tau_{r_1(m_1), r_2(m_2)}$ for which all prime factors of $m_1$ and $m_2$ lie in $\mathbb{P}$.

Both $G_C$ and $G_T$ are subgroups of $CT_{\{2,3\}}(\mathbb{Z})$. 

2010 Mathematics Subject Classification. 20B22, 11B99.
Remark 1.4. The group $CT_{\{2\}}(\mathbb{Z})$ is isomorphic to Higman’s group $G_{2,1}$ defined in [3]. This finitely presented infinite simple group is usually treated in the literature under the name Thompson’s group $V$.

The isomorphism between $CT_{\{2\}}(\mathbb{Z})$ and Thompson’s group $V$ has been pointed out by John P. McDermott in response to the question of the author which known simple group the former group would be isomorphic to.

If $|P| > 1$, the group $CT_P(\mathbb{Z})$ has no underlying tree structure. This makes the situation notably more complicated. Anyway if $P$ is finite, then $CT_P(\mathbb{Z})$ is still finitely generated – cf. Theorem 3.2.

2. A permutation group equivalent of the Collatz conjecture

In this section we prove Proposition 1.2.

Proposition 2.1. Let $a := \tau_{1(2),4(6)}$, $b := \tau_{1(3),2(6)}$ and $c := \tau_{2(3),4(6)}$. Then the group $G_C := \langle a, b, c \rangle < CT(\mathbb{Z})$ acts transitively on $\mathbb{N} \setminus 0(6)$ if and only if the Collatz conjecture holds.

Proof. We observe that $C^{-1}(0(3)) = 0(6) \subset 0(3)$, that the restrictions of $C$ and $a$ to $3(6)$ are the same and map this residue class to $10(18) \subset \mathbb{Z} \setminus 0(3)$, that $10(18) = 3(6)$, and that no trajectory of $C$ contains only multiples of 3. Therefore it suffices to show that for any $n \in \mathbb{N} \setminus 0(3)$ we have $\{n, na, nb, nc\} = \{n\} \cup \{n^C\} \cup C^{-1}(n)$. We treat four cases:

<table>
<thead>
<tr>
<th>$n \mod 6$</th>
<th>$n$</th>
<th>$n^a$</th>
<th>$n^b$</th>
<th>$n^c$</th>
<th>$n^C$</th>
<th>$C^{-1}(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$n$</td>
<td>$3n + 1$</td>
<td>$2n$</td>
<td>$n$</td>
<td>$3n + 1$</td>
<td>${2n}$</td>
</tr>
<tr>
<td>2</td>
<td>$n$</td>
<td>$n$</td>
<td>$\frac{n}{2}$</td>
<td>$2n$</td>
<td>$\frac{n}{2}$</td>
<td>${2n}$</td>
</tr>
<tr>
<td>4</td>
<td>$n$</td>
<td>$\frac{n-1}{2}$</td>
<td>$2n$</td>
<td>$\frac{n}{2}$</td>
<td>$\frac{n}{2}$</td>
<td>${\frac{n-1}{2}, 2n}$</td>
</tr>
<tr>
<td>5</td>
<td>$n$</td>
<td>$3n + 1$</td>
<td>$n$</td>
<td>$2n$</td>
<td>$3n + 1$</td>
<td>${2n}$</td>
</tr>
</tbody>
</table>

hence the proposition is proved. □

With a little more effort, we can get rid of the set $0(6)$ of fixed points:

Proposition 2.2. Let $a := \tau_{0(2),1(2)}$, $b := \tau_{1(2),2(4)}$ and $c := \tau_{1(4),2(6)}$. Then the group $G_T := \langle a, b, c \rangle < CT(\mathbb{Z})$ acts transitively on $\mathbb{N}_0$ if and only if the Collatz conjecture holds.

Proof. Let

$$T : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \begin{cases} \frac{n}{2} & \text{if } n \in 0(2), \\ \frac{3n+1}{2} & \text{if } n \in 1(2), \end{cases}$$

be the Collatz mapping, and put

$$f : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \begin{cases} n^{ac} = \frac{3n+4}{2} & \text{if } n \in 0(4), \\ n^{c} = \frac{3n+1}{2} & \text{if } n \in 1(4), \\ n^{b} = \frac{n}{2} & \text{if } n \in 2(4), \\ n^{aba} = \frac{n-3}{2} & \text{if } n \in 3(4), \end{cases}$$

and

$$r : \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \begin{cases} 2n - 2 & \text{if } n \in 0(3) \cup 2(3), \\ 2n - 1 & \text{if } n \in 1(3). \end{cases}$$
Then $rf$ and $Tr$ coincide on $\mathbb{Z} \setminus 0(6)$, and we have $rf^2 = T^2r$. Further, $a$ interchanges the image of $r$ with its complement in $\mathbb{Z}$. Therefore if the Collatz conjecture holds, then the group $G_T$ acts transitively on $\mathbb{N}_0$. It remains to show the other direction. Put

$$s : \mathbb{Z} \to \mathbb{Z}, \ n \mapsto \begin{cases} \frac{n+2}{2} & \text{if } n \in 0(2), \\ \frac{n+1}{2} & \text{if } n \in 1(2). \end{cases}$$

The mapping $s$ is a right inverse of $r$, and for all integers $n$ we have $n^s = n^{-s}$. It suffices to check that for all $n \in \mathbb{N}_0$ we have \{n^{bs}, n^{cs}\} = \{n^s, n^{sT}\} \cup T^{-1}(n^s)$. Indeed we have

- \(n^{bs} = n^s\) if \(n \in 0(4)\),
- \(n^{bs} = n^{sT}\) if \(n \in 2(4)\),
- \(n^{bsT} = n^s\) if \(n \in 1(2)\),
- \(n^{cs} = n^s\) if \(n \in 3(4) \cup 0(6) \cup 4(6)\),
- \(n^{cs} = n^{sT}\) if \(n \in 1(4)\), and
- \(n^{cs} = n^s\) if \(n \in 2(6)\),

which shows that if $G_T$ acts transitively on $\mathbb{N}_0$, then the Collatz conjecture holds. \(\Box\)

Note however that for some groups generated by three class transpositions it is easy to find out that they act transitively on $\mathbb{N}_0$:

**Remark 2.3.** With the GAP [2] package RCWA [6], using Method 10.4 in [4] one can check that the group $G_5 := \langle \tau_{0(2),1(2)}, \tau_{1(2),2(4)}, \tau_{(0,2),2(3)} \rangle$ acts at least 5-transitively on $\mathbb{N}_0$. The group $G_5$ can be obtained from $G_T$ by replacing the generator $\tau_{1(4),2(6)}$ by $\tau_{0(3),2(3)}$. The important difference between $G_5$ and $G_T$ is as follows: while there is a finite set $S$ of elements of $G_5$ such that for every integer $n > 0$ there is some $g \in S$ such that $n^g < n$, the group $G_T$ does not have a finite subset with this property.

### 3. Thompson’s Group $V$ and Further Subgroups of $CT(\mathbb{Z})$

By Theorem 2.3 in [5], the group $CT(\mathbb{Z})$ is not finitely generated. By the arguments used in the proof of that theorem, it follows also that $CT_{\mathbb{P}}(\mathbb{Z})$ is not finitely generated if $\mathbb{P}$ is infinite. However we will see that $CT_{\mathbb{P}}(\mathbb{Z})$ is finitely generated if $\mathbb{P}$ is finite.

**Definition 3.1.** Given a positive integer $m$, let $C_m$ be the set of all class transpositions which interchange residue classes whose moduli divide $m$.

**Theorem 3.2.** Let $\mathbb{P}$ be a finite set of primes. Then the group $CT_{\mathbb{P}}(\mathbb{Z})$ is finitely generated. More precisely, $CT_{\mathbb{P}}(\mathbb{Z})$ is generated by $C_m$, where $m := \prod_{p \in \mathbb{P}} p^2$ if $2 \notin \mathbb{P}$ and $m := 2 \cdot \prod_{p \in \mathbb{P}} p^2$ otherwise.

**Proof.** Let $m$ be as above, and let $\tau = \tau_{r_1(m_1), r_2(m_2)} \in CT_{\mathbb{P}}(\mathbb{Z})$ be a class transposition. We need to show that $\tau$ can be written as a product of elements of $C_m$.

Let $p \in \mathbb{P}$, and let $k_1$ and $k_2$ be the exponents of the highest powers of $p$ which divide $m_1$ or $m_2$, respectively. Without loss of generality, we can assume $k_2 \geq k_1$ and $k_2 > 2$.

We put $m_3 := \gcd(m_1, m_2)$ and $m_4 := m_3/p$. Since $r_1(m_1)$ and $r_2(m_2)$ are disjoint residue classes and $m_4 \geq 3$, we can choose a residue class $r_4(m_4)$ which intersects trivially with the support of $\tau$. Putting $\sigma := \tau_{r_2(m_3), r_4(m_4)} \in C_m$, we have $\tau^\sigma = \tau_{r_1(m_1), r_4(m_2)/p}$. Now we can conclude by induction on $k_i, i = 1, 2$, carried out for all primes $p \in \mathbb{P}$, that there is a product $\pi$ of elements of $C_m$ such that $\tau^\pi \in C_m$. The assertion follows. \(\Box\)

Small generating sets for the groups $CT_{\{2\}}(\mathbb{Z}) \cong G_{2.1}$ and $CT_{\{3\}}(\mathbb{Z})$ are immediate, and from Theorem 3.2, by means of computation with the GAP [2] package RCWA [6] we can also derive one for $CT_{\{2,3\}}(\mathbb{Z})$:
Proposition 3.3. We have
\[ \text{CT}_{\{2\}}(\mathbb{Z}) = \langle \tau_{0(2)}, \tau_{1(2)}, \tau_{0(2)}, \tau_{1(4)}, \tau_{0(2)}, \tau_{1(2)}, \tau_{1(4)} \rangle, \]
\[ \text{CT}_{\{3\}}(\mathbb{Z}) = \langle \tau_{0(3)}, \tau_{1(3)}, \tau_{2(3)}, \tau_{5(9)}, \tau_{7(9)} \rangle, \]
\[ \text{CT}_{\{2,3\}}(\mathbb{Z}) = \langle \tau_{0(2)}, \tau_{1(2)}, \tau_{0(3)}, \tau_{1(3)}, \tau_{2(3)}, \tau_{0(2)}, \tau_{1(4)}, \tau_{0(2)}, \tau_{3(4)} \rangle. \]

The generators for \( \text{CT}_{\{2\}}(\mathbb{Z}) \) given in Proposition 3.3 correspond directly to Higman's generators for \( G_{2,1} \):

Remark 3.4. As one can check by straightforward calculation, the generators \( \kappa := \tau_{0(2)}, \lambda := \tau_{1(2)}, \mu := \tau_{0(2)}, \) and \( \nu := \tau_{1(4)} \) for \( \text{CT}_{\{2\}}(\mathbb{Z}) \) given in Proposition 3.3 satisfy the following defining relations of the group \( G_{2,1} \) given in Higman [3], p. 50.:

(1) \( \kappa^2 = \lambda^2 = \mu^2 = \nu^2 = 1 \),
(2) \( \lambda \kappa \mu \kappa \mu \kappa \mu = 1 \),
(3) \( \kappa \nu \lambda \kappa \mu \lambda \nu \mu \kappa \mu = 1 \),
(4) \( (\lambda \kappa \mu \nu)^3 = (\mu \kappa \lambda \mu \nu)^3 = 1 \),
(5) \( (\lambda \nu \mu)^2 \kappa (\mu \nu \lambda)^2 \kappa = 1 \),
(6) \( (\lambda \nu \mu)^5 = 1 \),
(7) \( (\lambda \kappa \kappa \nu \lambda \nu \kappa \mu \nu)^3 \kappa \nu \kappa = 1 \),
(8) \( (\lambda \kappa \mu \nu \lambda \nu \kappa \mu \nu)^3 = 1 \),
(9) \( (\lambda \nu \kappa \kappa \mu \nu \kappa \mu \nu)^4 = 1 \),
(10) \( (\mu \kappa \kappa \lambda \nu \kappa \lambda \nu \kappa \mu \nu)^4 = 1 \),
(11) \( (\lambda \nu \kappa \kappa \mu \nu \kappa \mu \nu)^2 = 1 \), and
(12) \( (\mu \kappa \kappa \lambda \nu \kappa \mu \nu)^2 = 1 \)

Since \( G_{2,1} \) is simple, it follows that \( \text{CT}_{\{2\}}(\mathbb{Z}) \cong G_{2,1} \). Another presentation for this group can be found on Page 242 in [1]. The generators \( A, B, C \) and \( \pi_0 \) used there can be related to \( \kappa, \lambda, \mu \) and \( \nu \) via \( A = \lambda \kappa \mu, B = \mu \nu \lambda \kappa, C = \mu \kappa \lambda \kappa \) and \( \pi_0 = \mu \), respectively, \( \kappa = AC, \lambda = AC \pi_0 A^{-1}, \mu = \pi_0 \) and \( \nu = A \pi_0 B^{-1} \pi_0 \). The group \( \text{CT}_{\{2\}}(\mathbb{Z}) \) can be visualized as shown in Figure 1.

\[ \begin{array}{c}
\text{Z} \\
\downarrow \\
0(2) \quad \kappa \quad 1(2) \\
\downarrow \\
0(4) \quad 2(4) \quad 1(4) \quad 3(4) \\
\downarrow \\
0(8) \quad 4(8) \quad 2(8) \quad 6(8) \quad 1(8) \quad 5(8) \quad 3(8) \quad 7(8)
\end{array} \]

Figure 1. The arrows point to the roots of the subtrees interchanged by the generators.
References

3. Graham Higman, Finitely presented infinite simple groups, Notes on Pure Mathematics, Department of Pure Mathematics, Australian National University, Canberra, 1974. MR 0376874 (51 #13049)

E-mail address: stefan@gap-system.org