SIMPLE GROUPS THE DERIVED SUBGROUPS OF ALL OF WHOSE SUBGROUPS ARE TI-SUBGROUPS

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Abstract. We show that a non-abelian finite simple group the derived subgroups of all of its subgroups are TI-subgroups is isomorphic to either $\mathrm{PSL}(2, 2^p)$ for some prime p, to $\mathrm{PSL}(2, 7)$ or to the Suzuki group $\mathrm{Sz}(8)$.

1. Introduction

Recall that a group is said to be a *Dedekind group* if all of its subgroups are normal. As already Dedekind [4] himself has found, there are not many possibilities for the structure of such group. Therefore it is natural to weaken the condition of all subgroups being normal a bit, and to see whether one can still obtain a classification of the groups which satisfy such weakened condition.

An example of a property of a subgroup which is weaker than normality is that of being a TI-subgroup. – Recall that a subgroup is said to be a *TI-subgroup* if its distinct conjugates have pairwise trivial intersection. The groups all of whose subgroups are TI-subgroups can still be classified – cf. Walls [15].

Now it seems natural to further weaken the condition to a certain extent, and to classify groups a certain subset of whose subgroups are TI-subgroups. One result in this spirit is the classification of finite groups all of whose abelian subgroups are TI-subgroups obtained by Guo et al. [6]. Another is the description of the structure of the non-nilpotent groups all of whose cyclic subgroups are TI-subgroups obtained by Mousavi et al. [13], and the extension of this work to finite nilpotent groups by Abdollahi and Mousavi [1]. Here we are interested in another set of subgroups to be TI-subgroups:

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Definition 1.1. Let G be a group. We call a subgroup of G a *DTI-subgroup* if its derived subgroup is a TI-subgroup. Further, we call the group G a *DTI-group* if all of its subgroups are DTI-subgroups.

In this paper, using the CFSG we classify the finite simple DTI-groups. Our main result is the following:

Theorem 1.2. *Up to isomorphism, the non-abelian finite simple DTI-groups are precisely the following:*

(1) $\mathrm{PSL}(2, 2^p)$ *for some prime number* p*;*
(2) $\mathrm{PSL}(2, 7)$*;*
(3) $\mathrm{Sz}(8)$*.*

2. Preliminaries

For the obvious reasons, a subgroup whose derived subgroup has prime order is always a DTI-subgroup. Basic examples of DTI-groups are e.g. the dihedral groups, which even normalize the derived subgroups of all of their subgroups. A slightly more elaborate example is the following:

Example 2.1. The alternating group A_5 is a DTI-group. – Up to isomorphism, there are four non-trivial groups which occur as derived subgroups of subgroups of A_5, namely A_5, C_5, C_3 and V_4. Since A_5 normalizes itself and since subgroups of prime order are always TI-subgroups, it suffices to convince oneself that the conjugates of V_4 in A_5 have pairwise trivial intersection. – Note however that not all subgroups of A_5 are TI-subgroups. So for example any two distinct conjugates of $\mathrm{A}_4 < \mathrm{A}_5$ intersect in a cyclic group of order 3.

Lemma 2.2. *The following hold:*

(1) *A perfect subgroup is a DTI-subgroup if and only if it is a TI-subgroup.*
(2) *Every subgroup of a DTI-group is a DTI-group.*

Proof. Immediate. □

Lemma 2.3. *Characteristic subgroups of TI-subgroups are TI-subgroups as well.*

Proof. Let G be a group, and assume that H is a TI-subgroup of G and that K is a characteristic subgroup of H. Let $g \in G$. In case $H \cap H^g = 1$ we also have $K \cap K^g = 1$. Otherwise it is $H \cap H^g = H$, and the inner automorphism of H induced by conjugation with g fixes K – i.e. we have $K \cap K^g = K$. □

Lemma 2.4. *Let G be a non-abelian finite simple DTI-group, and let $M < G$ be a maximal subgroup. Then M is not simple, and we have $\mathrm{N}_G(M') = M$.*

Proof. Put $X := \{H' \mid H < G\}$. First we show that for every $H' \in X$ we have $H' \lneq \mathrm{N}_G(H')$. Clearly $\{1, G\} \subset X$, and X has at least three elements – for if $X = \{1, G\}$, then all proper subgroups of G are abelian and by [12], G is solvable, which contradicts our assumptions. If there exists $1 \neq H' \in X$ such that for every $g \in G \setminus H'$ we have $H' \cap H'^g = 1$, then G is a Frobenius group with complement H' – which contradicts the assumed simplicity of G. Hence for every $1 \neq H' \in X$ there exists $g \in G \setminus H'$ such that $H' \cap H'^g \neq 1$, i.e. $H' \cap H'^g = H'$. But this means that $g \in \mathrm{N}_G(H') \setminus H'$, and therefore $H' \lneq \mathrm{N}_G(H')$.

On the other hand $M' \neq 1$, since every finite group having an abelian maximal subgroup is solvable (see e.g. [7]). Now assuming $M = M'$ we would have $M' = M \lneq \mathrm{N}_G(M')$, and by maximality of M it would follow $\mathrm{N}_G(M') = G$ – which is impossible, since G is simple. Therefore it follows that $1 < M' < M$, and that M is not simple. □

By Lemma 2.4, one way to disprove a non-abelian simple group to be a DTI-group is to find a maximal subgroup which is simple.

On the other hand, by Lemma 2.2, a group which has a non-DTI-subgroup cannot be a DTI-group itself. By [2] every non-abelian simple group has a minimal simple subgroup, so our starting point here are the minimal simple groups. The latter are the non-abelian simple groups all of whose proper subgroups are solvable, or equivalently the non-abelian simple groups which no other non-abelian simple group embeds into. Thompson [14] determined all minimal simple groups:

Theorem 2.5. *The minimal simple groups are as follows:*

1. $\mathrm{PSL}(2, 2^p)$ *for a prime* p.
2. $\mathrm{PSL}(2, 3^p)$ *for an odd prime* p.
3. $\mathrm{PSL}(2, p)$ *for a prime* $p > 3$ *with* $p^2 + 1$ *divisible by* 5.
4. $\mathrm{PSL}(3, 3)$.
5. $\mathrm{Sz}(2^p)$ *for an odd prime* p.

3. The projective special linear groups $\mathrm{PSL}(2, q)$

In this section we prove that the projective special linear group $\mathrm{PSL}(2, q)$ is a DTI-group if and only if $q = 2^r$ where r is a prime number or $q = 7$. For this purpose, we need some properties of $\mathrm{PSL}(2, q)$ which we repeat here. These facts can be found in [9].

The group $\mathrm{PSL}(2, q)$ acts doubly transitively on the projective line $\mathbb{P}^1(\mathbb{F}_q)$. By using this action, the following result can be proved.

Lemma 3.1. *Let $q = p^n$ be a prime power. Then the following hold:*

(1) *The Sylow p-subgroups of* $\mathrm{PSL}(2, q)$ *are TI-subgroups.*
(2) *The cyclic subgroups of* $\mathrm{PSL}(2, q)$, *$q \neq 2$ having order prime to p are TI-subgroups*

The subgroups of $\mathrm{PSL}(2, q)$ are known by a Theorem of Dickson [5]. A complete list of subgroups of $\mathrm{PSL}(2, q)$ is as follows:

Theorem 3.2. *Let $q := p^n$ where p is a prime and $n \in \mathbb{N}$. Up to isomorphism, a complete list of subgroups of* $\mathrm{PSL}(2, q)$ *is as follows:*

(1) *Elementary abelian p-groups.*
(2) *Cyclic groups of order d, where $d \mid (q \pm 1)/\gcd(q - 1, 2)$.*
(3) *Dihedral groups of order $2d$, with d as in (2).*
(4) *Symmetric groups S_4 if $16 \mid q^2 - 1$.*
(5) *Alternating groups A_5 if $5 \mid q^2 - 1$ or $p = 5$.*
(6) *Alternating groups A_4 if $p > 2$ or $p = 2$ and $2 \mid n$.*
(7) *Semidirect products $\mathrm{C}_p^m \rtimes \mathrm{C}_t$ of elementary abelian groups of order p^m $(m \leq n)$ with cyclic groups of order t, where t divides $p^m - 1$ as well as $p^n - 1$.*
(8) *Groups $\mathrm{PSL}(2, p^m)$ for divisors m of n, and $\mathrm{PGL}(2, p^m)$ for divisors m of $n/2$ in case n is even.*
(9) *Subgroups of the above groups.*

From Theorem 3.2, we immediately obtain:

Lemma 3.3. *Let $q := p^r$ where p and r are prime numbers. Then up to isomorphism the derived subgroups of subgroups of* $\mathrm{PSL}(2, q)$ *are as follows:*

(1) *Cyclic groups of order d, where $d \mid (q \pm 1)/\gcd(q - 1, 2)$.*
(2) *Alternating groups A_4, provided that $16 \mid q^2 - 1$.*
(3) *Elementary abelian groups C_p^n for some $n \in \mathbb{N}$.*
(4) *The Klein four group V_4, if $p > 2$ or $p = 2$ and $r = 2$ (note that in the latter case, $q = 4$ and $G \cong \mathrm{A}_5$).*

By Lemma 3.1, the subgroups of Type (1) in Lemma 3.3 are TI-subgroups of $\mathrm{PSL}(2, q)$. In the sequel, we need the following elementary lemma:

Lemma 3.4. *Let a and b be positive integers, and put $c := \gcd(a, b)$. Then the following hold:*

(1) $\gcd(2^a + 1, 2^b + 1) = 2^c + 1$ *if a/c and b/c are both odd, and 1 otherwise;*
(2) $\gcd(2^a + 1, 2^b - 1) = 2^c + 1$ *if a/c is odd and b/c is even, and 1 otherwise;*

(3) $\gcd(2^a-1, 2^b-1) = 2^c-1$.

Proof. Immediate. □

Lemma 3.5. *Put $G := \mathrm{PSL}(2, 2^r)$, $r \in \mathbb{N}$. Then G is a DTI-Group if and only if r is a prime number.*

Proof. First assume that r is composite. Let s be a prime divisor of r. Then G has a subgroup isomorphic to $\mathrm{PSL}(2, 2^{r/s})$, which is maximal by Theorem 3.2. Hence in this case G has a maximal subgroup which is simple, and is therefore not a DTI-group by Lemma 2.4.

Now assume that r is prime. Clearly $16 \nmid (2^r)^2 - 1$, so G has no subgroup whose derived subgroup is of Type (2) of Lemma 3.3. Also the derived subgroups of the groups of Type (3) in Theorem 3.2 are TI-subgroups of G. So it is enough to show that elementary abelian subgroups C_2^n, $n \in \mathbb{N}$, are TI-subgroups of G.

The Sylow 2-subgroups of G are elementary abelian of order 2^r. Let P be a Sylow 2-subgroup of G. Then the normalizer $\mathrm{N}_G(P)$ acts transitively on $P \setminus \{1\}$ by conjugation. Thus if $Q < P$ is a subgroup of order 2^n, then Q can only be a TI-subgroup if $2^n - 1$ divides $2^r - 1$. Since r is prime, this only holds for $n = 1$ and $n = r$. On the other hand, for any $1 < n < r$ we have $\gcd(2^n - 1, 2^r - 1) = 1$, so G has no subgroup of type $\mathrm{C}_2^n \rtimes \mathrm{C}_t$. Thus, the cyclic subgroups of order 2 and the Sylow 2-subgroups are the only elementary abelian 2-subgroups of G which occur as derived subgroups of suitable subgroups of G.

Conversely since $\gcd(2^n - 1, 2^r - 1) = 1$ for every $n < r$, the only subgroups of Type (7) of Theorem 3.2 are subgroups $\mathrm{C}_2^r \rtimes \mathrm{C}_t$, where $t \mid 2^r - 1$. In fact, C_2^r is a Sylow 2-subgroup of G. Therefore by Lemma 3.1 it is a TI-subgroup of G. □

Lemma 3.6. *Put $G := \mathrm{PSL}(2, q)$ for some prime power q satisfying $16 \mid q^2 - 1$. Then G is a DTI-group if and only if $q = 7$.*

Proof. Let q be a prime power such that $16 \mid q^2 - 1$, and assume that $G := \mathrm{PSL}(2, q)$ is a DTI-group. By [5], the normalizer in G of every subgroup of G which is isomorphic to A_4 is isomorphic to S_4, and every subgroup of A_4 of order 3 is self-normalizing. Also A_4 has 4 maximal subgroups of order 3, all conjugate. The key is to count the subgroups of G isomorphic to A_4 which contain a given cyclic subgroup C_3 of order 3. – We have $\mathrm{N}_G(\mathrm{C}_3) \cong \mathrm{D}_{q-1}$ if $3 \mid q - 1$, and $\mathrm{N}_G(\mathrm{C}_3) \cong \mathrm{D}_{q+1}$ if $3 \mid q + 1$ – where D_{q+1} and D_{q-1} denote the dihedral groups of orders $q + 1$ and $q - 1$, respectively. Thus in case $3 \mid q - 1$, the number of subgroups of G isomorphic to A_4 which contain C_3 is equal to

$$\frac{|\operatorname{PSL}(2,q)|}{|\operatorname{S}_4|} \times 4 \times \frac{q-1}{|\operatorname{PSL}(2,q)|} = \frac{q-1}{6}.$$

Similarly, in case $3 \mid q+1$ the number of such subgroups is $(q+1)/6$. Now since G is a DTI-group, A_4 must be a TI-subgroup. As G has two conjugacy classes of subgroups isomorphic to A_4, this implies that $(q-1)/6, (q+1)/6 \in \{1,2\}$, and hence $q \in \{5,7,11,13\}$. Now our initial assumption $16 \mid q^2-1$ yields $q=7$. Checking that $\operatorname{PSL}(2,7)$ is indeed a DTI-group completes the proof. □

Lemma 3.7. *Put $G := \operatorname{PSL}(2,q)$, where $q = p^m \geq 9$ is an odd prime power. If $16 \nmid q^2-1$, then G is not a DTI-group.*

Proof. By Theorem 3.2, the group G has a subgroup isomorphic to A_4. We show that the derived subgroup of the latter, which is isomorphic to V_4, is not a DTI-subgroup of G. By [5], G has a single class of $q(q^2-1)/24$ conjugate subgroups isomorphic to V_4. On the other hand, each of these groups has 3 involutions. Now in total, the group G has $q(q+t)/2$ involutions, where $t \in \{1,-1\}$ and $4 \mid (q-t)$. The assumption that the conjugate copies of V_4 intersect trivially finally implies that

$$\frac{3q(q^2-1)}{24} = \frac{q(q \pm 1)}{2},$$

hence $(q^2-1)/4 = q \pm 1$, which contradicts our assumption $q \geq 9$. □

Corollary 3.8. *The minimal simple groups of the form $\operatorname{PSL}(2,3^r)$, where r is a prime, are not DTI-groups.*

Corollary 3.9. *Let $G := \operatorname{PSL}(2,q)$ be a minimal simple group. Then G is a DTI-group if and only if $q = 2^r$ for some prime r or $q = 7$.*

4. The Suzuki groups

In the following we collect some results on the subgroups of Suzuki groups. The corresponding proofs can be found in [[8], Thm. 3.10 in Ch. XI] and [[10], Thm. 4.12].

Theorem 4.1. *Let $G := \operatorname{Sz}(q)$, where $q = 2^{2m+1}$ for some positive integer m. Then a complete list of the subgroups of G is as follows:*

1. *The Hall subgroup $\mathrm{N}_G(T) = TH$ which is a Frobenius group of order $q^2(q-1)$, where T is a Sylow 2-subgroup of G.*
2. *The dihedral group $T_0 = \mathrm{N}_G(H)$ of order $2(q-1)$.*
3. *The cyclic Hall subgroups A_1 and A_2, where $|A_1| = 2^{2m+1} + 2^{m+1} + 1$ and $|A_2| = 2^{2m+1} - 2^{m+1} + 1$.*

(4) *The Frobenius subgroups* $T_1 = \mathrm{N}_G(A_1)$ *and* $T_2 = \mathrm{N}_G(A_2)$, *where* $|T_1| = 4|A_1|$ *and* $|T_2| = 4|A_2|$.
(5) *The subgroups of the form* $\mathrm{Sz}(r)$ *where* r *is an odd power of* 2, $r \geq 8$ *and* $q = r^n$ *for some* $n \in \mathbb{N}$.
(6) *Subgroups (and their conjugates) of the above groups.*

Moreover the conjugates of T, H, A_1 *and* A_2 *form a partition of* G. *So they are TI-subgroups of* G.

Lemma 4.2. *Let* p *be an odd prime number. Then the Suzuki group* $\mathrm{Sz}(2^p)$ *is a DTI-group if and only if* $p = 3$.

Proof. Put $G := \mathrm{Sz}(2^p)$. By Lemma 2.3 and Theorem 4.1 it suffices to check whether the 2-subgroups which occur as derived subgroups of subgroups of G are TI-subgroups. Suppose that $T \in \mathrm{Syl}_2(G)$, and that $Q < T$ is a 2-subgroup which occurs as derived subgroup of a subgroup of G. Assume $|Q| = 2^n$. Now $\mathrm{N}_G(T)$ acts transitively on the set $T \setminus \{1\}$. If Q is a TI-subgroup, then $Q \setminus \{1\}$ is a block under this action, and therefore we have $2^n - 1 \mid |T| - 1 = 2^{2p} - 1 = (2^p - 1)(2^p + 1)$. If $2^n - 1 \mid 2^p - 1$, then $n = p$ or $n = 1$. If $2^n - 1 \mid 2^p + 1$, then by Lemma 3.4 we have $\gcd(2^n - 1, 2^p + 1) = 3$ if n is even. So $n = 1$ or $n = 2$. This means that if a 2-subgroup Q of G is a TI-subgroup, then $|Q| \in \{2, 4, 2^p, 2^{2p}\}$. But in case $p > 3$, it is known that T always has a 3-generated subgroup whose derived subgroup is isomorphic to C_2^3, and which by the above cannot be a TI-subgroup. Finally, it is straightforward to handle the case $p = 3$, and to check that the group $G = \mathrm{Sz}(8)$ is indeed a DTI-group. $\square$

Lemma 4.3. *Let* m *be an odd composite number. Then the Suzuki group* $\mathrm{Sz}(2^m)$ *is not a DTI-group.*

Proof. Let p be a prime divisor of m. Then by Theorem 4.1, the group $\mathrm{Sz}(2^m)$ contains a maximal subgroup which is isomorphic to the simple group $\mathrm{Sz}(2^{m/p})$. Now the assertion follows from Lemma 2.4. $\square$

5. The simple groups which are not minimal

In this section we show that the non-minimal simple groups are not DTI-groups. For this purpose, we need the following lemma.

Lemma 5.1. *Let* q *be a prime power. Then* $\mathrm{SL}(3, q)$ *and* $\mathrm{PSL}(3, q)$ *are DTI-groups if and only if* $q = 2$.

Proof. We handle the cases $q = 2$ and $q = 3$ by means of computation, and assume from now on that $q \geq 4$. Let $G := \mathrm{SL}(3, q)$, and put

$$H \ := \ \left\{ \left(\begin{array}{c|c} h & 0 \\ \hline 0 & 1 \end{array} \right) \ \middle| \ h \in \mathrm{SL}(2, q) \right\} \ < \ G.$$

Then $H \cong \mathrm{SL}(2, q)$ is perfect. Now putting

$$t \ := \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$H \cap H^t \ = \ \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ \middle| \ a \in \mathbb{F}_q \right\} \ \cong (\mathbb{F}_q, +) \notin \{1, H\},$$

and therefore H is not a DTI-subgroup, and G is not a DTI-group. Now taking images under the canonical projection modulo the centre of G, all of this argumentation remains intact. – Homomorphic images of perfect groups are still perfect groups, and as the order of the centre is either 1 or 3 (hence in particular smaller than q), the intersection of the conjugates cannot be merged into one equivalence class. The assertion for the groups $\mathrm{PSL}(3, q)$ follows. □

Lemma 5.2. *Suppose that G is a non-abelian simple group which is not minimal simple. Then G is not a DTI-group.*

Proof. We use the classification of finite simple groups, cf. e.g. [3]. First we consider the sporadic simple groups. The table given below shows that every sporadic simple group has a proper non-DTI-subgroup, and that therefore there is no sporadic simple DTI-group:

Group	Non-DTI-subgroup	Group	Non-DTI-subgroup
M_{11}	$\mathrm{PSL}(2, 11)$	M_{12}	M_{11}
J_1	$\mathrm{PSL}(2, 11)$	M_{22}	A_7
J_2	$\mathrm{PSU}(3, 3)$	M_{23}	M_{11}
${}^2\mathrm{F}_4(2)'$	$\mathrm{PSL}(2, 25)$	HS	M_{22}
J_3	$\mathrm{PSL}(2, 19)$	M_{24}	M_{23}
McL	M_{22}	He	A_7
Ru	A_8	Suz	A_7
O′N	J_1	Co_3	HS
Co_2	McL	Fi_{22}	A_{10}
HN	A_{12}	Ly	A_{11}
			To be continued.

Continued.

Group	Non-DTI-subgroup	Group	Non-DTI-subgroup
Th	$\mathrm{PSL}(2,19)$	Fi_{23}	A_{12}
Co_1	Co_2	J_4	$\mathrm{PSL}(2,23)$
Fi'_{24}	Fi_{23}	B	M_{11}
M	Th		

Now suppose that G is a group of Lie type. If G is a Chevalley group except $\mathrm{PSL}(2,q)$ and $\mathrm{PSp}(4,q)$, then by [2], it has a subgroup isomorphic to either $\mathrm{SL}(3,q)$ or $\mathrm{PSL}(3,q)$. By Lemma 5.1 the latter are not DTI-groups unless $q=2$, and the case $q=2$ can be dealt with in a tedious case-by-case analysis. If $G=\mathrm{PSp}(4,q)$ and q is even, then by [11], it has a maximal subgroup isomorphic to the Suzuki group $\mathrm{Sz}(q)$ – and therefore it is not a DTI-group by Lemma 2.4.

If G is a Steinberg group, then it is of one of the types ${}^2\mathrm{A}_n(q^2)$, $n\geq 2$, ${}^2\mathrm{D}_n(q^2)$, $n\geq 4$, ${}^2\mathrm{E}_6(q^2)$ or ${}^3\mathrm{D}_4(q^3)$. Now again by [2], the groups ${}^2\mathrm{D}_n(q^2)$, $n\geq 4$, ${}^2\mathrm{E}_6(q^2)$ and ${}^3\mathrm{D}_4(q^3)$ have subgroups isomorphic to either $\mathrm{SL}(3,q)$ or $\mathrm{PSL}(3,q)$ – and again by Lemma 5.1 the latter are not DTI-groups unless $q=2$. Also, as above the case $q=2$ can be dealt with in a tedious case-by-case analysis. Further if $n\geq 5$, then ${}^2\mathrm{A}_n(q^2)$ has a proper subgroup isomorphic to either $\mathrm{SL}(3,q^2)$ or $\mathrm{PSL}(3,q^2)$. So we consider the Steinberg groups ${}^2\mathrm{A}_n(q^2)$ for $n\in\{2,3,4\}$. For q odd, the group ${}^2\mathrm{A}_n(q^2)$ has the simple subgroup $\mathrm{P}\Omega_n^+(q)$, which is not a DTI-group by the previous paragraph. So we may assume that q is a power of 2. Since the groups ${}^2\mathrm{A}_2(q^2)\cong\mathrm{PSL}(2,q)$ have already been treated before, we may assume that $n\in\{3,4\}$. But then the group ${}^2\mathrm{A}_n(q^2)$ has a subgroup isomorphic to the simple group $\mathrm{SL}(2,q^2)$, hence it is not a DTI-group.

The simple groups of types ${}^2\mathrm{F}_4(2^{2m+1})$ and ${}^2\mathrm{G}_2(3^{2n+1})$ have non-DTI-subgroups isomorphic to $\mathrm{PSL}(2,25)$ and $\mathrm{PSL}(2,3^{2n+1})$, respectively. Finally, in Section 4 we have shown that except for $\mathrm{Sz}(8)$ the Suzuki groups are not DTI-groups, and the proof is completed. □

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