Classifying finite simple groups with at most 17 orbits under automorphisms

Definition 1: Let G denote a finite group, $h(G)$ its class number and $\omega(G)$ the number of orbits on G under the action of $\mathrm{Aut}(G)$. The term 'finite simple group' should always be understood as 'non-abelian finite simple group'.

Theorem 1: For any finite simple group G we have

$$|\mathrm{Out}(G)| < {}_2\log|G| .$$

Remark 1: This bound is sharp in the sense that it cannot be improved by more than a small constant factor, e.g. replacing ${}_2\log|G|$ by $\ln|G|$ would make the assertion fail in infinitely many cases.

Theorem 2:

a) For any positive integer n, there is only a finite number of finite simple groups G satisfying $\omega(G) \leq n$.

b) If G is a finite simple group of Lie type of Lie rank l over the field $\mathrm{GF}(p^f)$, then we have

$$\omega(G) \geq \frac{h(G)}{|\operatorname{Out}(G)|} \geq \frac{p^{lf}}{6l(l+1)f} .$$

Theorem 4: For $n \in \mathbb{N}$ and a prime power $q = p^f$, we have

$$\omega(\mathrm{PSL}(n,q)) \geq \frac{q^{n-1}}{2(n,q-1)f} ,$$

hence we can omit the exponent 2 of $(n, q-1)$ in the denominator of the lower bound given in Theorem 3.

Theorem 3: Let $q := p^f$ be a prime power. Then we have the following bounds on $\omega(G)$ for the series of finite simple groups of Lie type:

a) $\omega(\mathrm{PSL}(n,q)) \geq \dfrac{q^{n-1}}{(2-\delta_{n,2})(n,q-1)^2 f}$,

$\omega(\mathrm{PSU}(n,q)) \geq \dfrac{q^{n-1}}{2(n,q+1)^2 f} \quad (n \geq 3)$,

b) $\omega(\mathrm{O}(2n+1,q)) \geq \dfrac{q^n}{(1+\delta_{n,2}\delta_{p,2})(2,q-1)^2 f} \quad (n \geq 2)$,

$\omega(\mathrm{Sz}(q)) = \omega(\mathrm{PSL}(2,q)) + 2$,

c) $\omega(\mathrm{PSp}(2n,q)) \geq \dfrac{q^n}{(2,q-1)^2 f} \quad (n \geq 3)$,

d) $\omega(\mathrm{O}^+(2n,q)) \geq \dfrac{q^n}{(4,q^n-1)(2,q-1)^2(2+4\delta_{n,4}) f} \quad (n \geq 4)$,

$\omega(\mathrm{O}^-(2n,q)) \geq \dfrac{q^n}{2(4,q^n+1)^2 f} \quad (n \geq 4)$, $\omega({}^3\mathrm{D}_4(q)) \geq \dfrac{q^4}{3f}$,

e) $\omega(\mathrm{G}_2(q)) \geq \dfrac{q^2}{(1+\delta_{p,3}) f}$, $\omega({}^2\mathrm{G}_2(q)) = \omega(\mathrm{Ree}(q)) \geq \dfrac{q}{f}$,

f) $\omega(\mathrm{F}_4(q)) \geq \dfrac{q^4}{(1+\delta_{p,2}) f}$, $\omega({}^2\mathrm{F}_4(q)) = \omega(\mathrm{Ree}(q)) \geq \dfrac{q^2}{f}$,

g) $\omega(\mathrm{E}_6(q)) \geq \dfrac{q^6}{2(3,q-1)^2 f}$, $\omega({}^2\mathrm{E}_6(q)) \geq \dfrac{q^6}{2(3,q+1)^2 f}$,

$\omega(\mathrm{E}_7(q)) \geq \dfrac{q^7}{(2,q-1)^2 f}$, $\omega(\mathrm{E}_8(q)) \geq \dfrac{q^8}{f}$.

We define

$$\omega(l,p,f) \;:=\; \frac{p^{lf}}{6l(l+1)f}.$$

The partial derivatives of $\omega(l,p,f)$ satisfy

$$\frac{\partial\omega}{\partial l}\,\omega(l,p,f) \;=\; \frac{p^{lf}(l(l+1)f\ln p - 2l - 1)}{6l^2(l+1)^2 f}$$

$$> \; 0$$

$$\text{if} \quad l \geq 3 \text{ or } p \geq 5 \text{ or } f \geq 3,$$

$$\frac{\partial\omega}{\partial p}\,\omega(l,p,f) \;=\; \frac{p^{lf-1}}{6(l+1)} \;>\; 0$$

$$\text{for all} \quad l, p, f,$$

$$\frac{\partial\omega}{\partial f}\,\omega(l,p,f) \;=\; \frac{p^{lf}(lf\ln p - 1)}{6l(l+1)f^2} \;>\; 0$$

$$\text{if} \quad l \geq 2 \text{ or } p \geq 3 \text{ or } f \geq 2,$$

and $\omega(l,p,f) < 2$ for all $2^3 = 8$ triples with $l \leq 2$, $p \leq 3$ and $f \leq 2$, hence if we assume $\omega_{\max} \geq 2$, we can get our triples by the following GAP- function:

```
AdmissibleTriples := function ( OmegaMax )

  local  Try, sol;

  Try := function ( l, p, f )

    if      p^(l*f)/(6*l*(l+1)*f)
          <= OmegaMax
         and not [l,p,f] in sol
    then
      Add(sol,[l,p,f]);
      Try(l,p,f+1);
      Try(l,NextPrimeInt(p),f);
      Try(l+1,p,f);
    fi;
  end;

  sol := [];
  Try(1,2,1);
  return Set(sol);
end;
```

$PSL(2,4)$	$PSL(2,8)$	$PSL(2,16)$	$PSL(2,32)$	$PSL(2,64)$
$PSL(2,9)$	$PSL(2,27)$	$PSL(2,81)$	$PSL(2,5)$	$PSL(2,25)$
$PSL(2,7)$	$PSL(2,49)$	$PSL(2,11)$	$PSL(2,13)$	$PSL(2,17)$
$PSL(2,19)$	$PSL(2,23)$	$PSL(2,29)$	$PSL(2,31)$	$PSL(3,2)$
$PSL(3,4)$	$PSL(3,8)$	$PSL(3,16)$	$PSL(3,3)$	$PSL(3,5)$
$PSL(3,7)$	$PSL(4,2)$	$PSL(4,4)$	$PSL(4,3)$	$PSL(4,5)$
$PSL(5,2)$	$PSL(6,2)$			
$PSU(3,4)$	$PSU(3,8)$	$PSU(3,32)$	$PSU(3,3)$	$PSU(3,5)$
$PSU(3,11)$	$PSU(3,17)$	$PSU(4,2)$	$PSU(4,4)$	$PSU(4,3)$
$PSU(4,5)$	$PSU(4,7)$	$PSU(5,2)$	$PSU(5,4)$	$PSU(6,2)$
$PSU(9,2)$				
$O(5,4)$	$O(5,8)$	$O(5,3)$	$O(5,9)$	$O(5,5)$
$O(5,7)$	$O(7,2)$	$O(7,3)$	$O(9,2)$	
$Sz(8)$	$Sz(32)$			
$PSp(6,2)$	$PSp(6,3)$	$PSp(8,2)$		
$O^+(8,2)$	$O^+(8,3)$	$O^+(8,5)$	$O^+(10,2)$	$O^+(10,3)$
$O^-(8,2)$	$O^-(8,3)$	$O^-(10,2)$	$O^-(10,3)$	
${}^3D_4(2)$				
$G_2(4)$	$G_2(3)$			
$Ree(27)$				
$F_4(2)$	${}^2E_6(2)$			

f	$\omega(\mathrm{PSL}(2,p^f)),\ p \neq 2$
1	$\frac{1}{2}(p+3)$
2	$\frac{1}{4}(p^2+2p+5)$
3	$\frac{1}{6}(p^3+2p+9)$
4	$\frac{1}{8}(p^4+2p^2+4p+9)$

The number of orbits of the character table automorphism group on the set of conjugacy classes as a lower bound on $\omega(G)$.

G	$\omega(G)\ \geq$	G	$\omega(G)\ \geq$
$\mathrm{PSL}(4,3)$	20	$\mathrm{O}^+(8,2)$	27
$\mathrm{PSU}(3,11)$	18	$\mathrm{O}^+(8,3)$	37
$\mathrm{PSU}(6,2)$	34	${}^3\mathrm{D}_4(2)$	21

The value $\frac{h(G)-1}{|\mathrm{Out}(G)|}+1$ as a lower bound on $\omega(G)$.

G	$h(G)$	$\lvert\mathrm{Out}(G)\rvert$	$\omega(G) \geq$
PSL(4, 4)	84	4	21
PSL(6, 2)	60	2	30
PSU(3, 17)	106	6	18
PSU(4, 4)	94	4	24
PSU(4, 5)	97	4	25
PSU(5, 2)	47	2	24
PSU(9, 2)	402	6	67
O(5, 5)	34	2	18
O(5, 7)	52	2	26
O(7, 2) $\cong$ PSp(6, 2)	30	1	30
O(7, 3)	58	2	30
O(9, 2)	81	1	81
PSp(6, 3)	74	2	38
PSp(8, 2)	81	1	81
$\mathrm{O}^+(10, 2)$	97	2	49
$\mathrm{O}^-(8, 2)$	39	2	20
$\mathrm{O}^-(8, 3)$	112	4	29
$\mathrm{O}^-(10, 2)$	115	2	58
$\mathrm{F}_4(2)$	95	2	48
${}^2\mathrm{E}_6(2)$	126	6	22

For the groups $PSU(3,32)$, $PSU(4,7)$, $O(5,8)$, $O(5,9)$, $O^+(8,5)$, $O^+(10,3)$ and $O^-(10,3)$, we get our bounds by

1. (pseudo-)randomly searching elements in the corresponding universal groups with as many as possible different characteristic polynomials,

2. computing the number of orbits on the set of all occuring characteristic polynomials under the action of field automorphisms, and

3. dividing this number by the product of the order of the centre of the group, the order of the group of diagonal automorphisms and the order of the group of graph automorphisms.

This yields the lower bounds 18, 23, 19, 21, 24, 50, resp. 28.

Further groups that need to be considered:

- The sporadic simple groups G having not more than $17 \cdot |\mathrm{Out}(G)| - 1$ conjugacy classes; these are M_{11}, M_{12}, M_{22}, M_{23} J_1, J_2, J_3, HS, McL, He, O'N and the Tits group. $\mathrm{Out}(G)$ is trivial for $G \in \{\mathrm{M}_{11}, \mathrm{M}_{23}, \mathrm{J}_1\}$, hence the orbit numbers equal the class numbers:

  $\omega(\mathrm{M}_{11}) = 10$, $\omega(\mathrm{M}_{23}) = 17$, $\omega(\mathrm{J}_1) = 15$.

  The other nine groups G have index 2 in their respective automorphism group. Here we get $\omega(G)$ as the cardinality of the pre-image of 1 under the non-trivial character of degree 1 of $\mathrm{Aut}(G)$:

  $\omega(\mathrm{M}_{12}) = 12$, $\omega(\mathrm{M}_{22}) = 11$, $\omega(\mathrm{J}_2) = 16$, $\omega(\mathrm{J}_3) = \omega({}^2\mathrm{F}_4(2)') = 17$, $\omega(\mathrm{HS}) = 21$, $\omega(\mathrm{McL}) = 19$, $\omega(\mathrm{He}) = 26$, $\omega(\mathrm{ON}) = 25$.

- The alternating groups A_n for $n \leq 9$. The group $\mathrm{A}_6 \cong \mathrm{PSL}(2,9)$ has already been considered before, and by counting partitions of n with an even number of even parts we get

  $\omega(\mathrm{A}_5) = 4$, $\omega(\mathrm{A}_7) = 8$, $\omega(\mathrm{A}_8) = 12$, and $\omega(\mathrm{A}_9) = 16$.

Values of $\omega(G)$ which have been computed by 'brute force'.

G	$\omega(G)$
$\mathrm{PSL}(3,4)$	6
$\mathrm{PSL}(3,8)$	17
$\mathrm{PSL}(3,16)$	20
$\mathrm{PSL}(3,5)$	19
$\mathrm{PSL}(3,7)$	16
$\mathrm{PSL}(4,5)$	34
$\mathrm{PSL}(5,2)$	20
$\mathrm{PSU}(3,4)$	9
$\mathrm{PSU}(3,8)$	10
$\mathrm{PSU}(3,3)$	10
$\mathrm{PSU}(3,5)$	10
$\mathrm{PSU}(4,2) \cong \mathrm{O}(5,3)$	15
$\mathrm{PSU}(4,3)$	14
$\mathrm{O}(5,4)$	12
$\mathrm{G}_2(3)$	17
$\mathrm{G}_2(4)$	24
$\mathrm{Ree}(27)$	19

n	Simple groups G satisfying $\omega(G) = n$
4	$\mathrm{PSL}(2,4) \cong \mathrm{PSL}(2,5) \cong \mathrm{A}_5$
5	$\mathrm{PSL}(2,7) \cong \mathrm{PSL}(3,2)$, $\mathrm{PSL}(2,9) \cong \mathrm{A}_6$, $\mathrm{PSL}(2,8)$
6	$\mathrm{PSL}(3,4)$
7	$\mathrm{PSL}(2,11)$, $\mathrm{PSL}(2,16)$, $\mathrm{PSL}(2,27)$, $\mathrm{Sz}(8)$
8	$\mathrm{PSL}(2,13)$, A_7
9	$\mathrm{PSL}(3,3)$, $\mathrm{PSL}(2,32)$, $\mathrm{PSU}(3,4)$
10	$\mathrm{PSL}(2,17)$, $\mathrm{PSU}(3,3)$, $\mathrm{PSL}(2,25)$, M_{11}, $\mathrm{PSU}(3,5)$, $\mathrm{PSU}(3,8)$
11	$\mathrm{PSL}(2,19)$, M_{22}, $\mathrm{Sz}(32)$
12	$\mathrm{PSL}(4,2) \cong \mathrm{A}_8$, M_{12}, $\mathrm{O}(5,4)$
13	$\mathrm{PSL}(2,23)$
14	$\mathrm{PSU}(4,3)$
15	$\mathrm{PSU}(4,2) \cong \mathrm{O}(5,3)$, J_1, $\mathrm{PSL}(2,64)$, $\mathrm{PSL}(2,81)$
16	$\mathrm{PSL}(2,29)$, A_9, J_2, $\mathrm{PSL}(3,7)$
17	$\mathrm{PSL}(2,31)$, $\mathrm{PSL}(2,49)$, $\mathrm{G}_2(3)$, M_{23}, $\mathrm{PSL}(3,8)$, ${}^2\mathrm{F}_4(2)'$, J_3