A REFORMULATION OF THE 3N+1 CONJECTURE IN TERMS OF A MAPPING FROM THE FREE MONOID OF RANK 2 TO THE POSITIVE INTEGERS

STEFAN KOHL

ABSTRACT. The $3n+1$ Conjecture asserts that iterated application of the Collatz mapping $T : \mathbb{Z} \to \mathbb{Z},\ n \mapsto (n/2$ if n even, $(3n+1)/2$ if n odd$)$ to any positive integer yields 1 after a finite number of steps.

We construct a mapping from the free monoid of rank 2 to the set of positive integers, which is bijective if and only if the $3n+1$ Conjecture holds. We also rewrite the $3n+1$ Conjecture as a conjugacy problem for transformation monoids.

1. INTRODUCTION

In the 1930s, Lothar Collatz made the following conjecture which is still open today:

Conjecture 1.1 (3n+1 Conjecture)**.** Iterated application of the mapping

$$T:\ \mathbb{Z} \longrightarrow \mathbb{Z},\ \ n\ \longmapsto \begin{cases} \frac{n}{2} & \text{if } n \text{ even,} \\ \frac{3n+1}{2} & \text{if } n \text{ odd} \end{cases}$$

to any positive integer yields 1 after a finite number of steps. In short this means that for all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}_0$ such that $T^{(k)}(n) = 1$.

For a survey article on this conjecture, we refer to [2].

As formulating a problem in a different way often helps in finding a solution, the $3n+1$ Conjecture has already been reformulated in a large number of ways. For details, see Lagarias' $3n+1$ problem annotated bibliography [3] and the publications indexed there.

In this article we construct a mapping from the free monoid of rank 2 to the set of positive integers, which is bijective if and only if the $3n+1$ Conjecture holds. We also rewrite the $3n+1$ Conjecture as a conjugacy problem for transformation monoids.

2. THE 3N+1 TREE

Obviously, the $3n+1$ Conjecture is equivalent to the assertion that for any $n \in \mathbb{N}$ there is a $k \in \mathbb{N}_0$ such that n lies in the preimage of 1 under $T^{(k)}$. This motivates the following definition:

Definition 2.1. Let $\mathcal{T}_0$ be the tree with root 8 in which the left child of a vertex n is $2n$, and in which the right child of a vertex $n \equiv 2 \bmod 3$ is $(2n-1)/3$. We call this infinite rooted tree the $3n+1$ tree.

In this notation, we ask whether all positive integers except of 1, 2 and 4 are vertices of $\mathcal{T}_0$. The reason for not taking 1 as the root of $\mathcal{T}_0$ is that due to the cycle $(1\ 2)$ of the Collatz mapping, $\mathcal{T}_0$ would not be cycle-free otherwise.

2000 *Mathematics Subject Classification.* Primary 11B99.

3. Turning the 3n+1 Tree Into a Complete Binary Tree

The automorphism group of the $3n+1$ tree is trivial. This holds since there is no vertex whose left and right subtree are isomorphic. Investigating more symmetric trees is usually easier. The most symmetric trees are the ones in which any vertex has the same number c of child vertices. For the sake of minimality, we choose $c = 2$ and try to turn the $3n + 1$ tree into a complete infinite binary tree.

For this purpose we cut away all vertices which either have no right child or whose right subtree is degenerate. This means that precisely the vertices remain that lie in the set $(2(9) \cup 8(9)) \cap \mathbb{N}$. Since any trajectory of the Collatz mapping contains a number not divisible by 3 and since we have

$$1(3) \cup 2(3) \;=\; \bigcup_{k=0}^{3} T^{(k)}(2(9) \cup 8(9)),$$

the $3n+1$ Conjecture is equivalent to the assertion that the vertices of the resulting tree $\mathcal{T}_1$ are precisely the numbers $n \in (2(9) \cup 8(9)) \cap \mathbb{N} \setminus \{2\}$.

Elementary calculations show that starting from some vertex n of $\mathcal{T}_1$, following the left respectively right branch means applying the injective mappings

$$l : n \mapsto \begin{cases} 4n & \text{if } n \in 2(9), \\ 16n & \text{if } n \in 8(9), \\ n & \text{otherwise}, \end{cases} \quad \text{or} \quad r : n \mapsto \begin{cases} \frac{2n-1}{3} & \text{if } n \in 17(27) \cup 26(27), \\ \frac{4n-2}{3} & \text{if } n \in 2(27) \cup 20(27), \\ \frac{8n-4}{3} & \text{if } n \in 8(27), \\ \frac{16n-8}{3} & \text{if } n \in 11(27), \\ n & \text{otherwise}, \end{cases}$$

respectively. Although our tree has only positive integers $n \in 2(9) \cup 8(9)$ as vertices, we take $\mathbb{Z}$ as the source of our mappings – this makes the notation easier, and does not cause any harm. The images of $2(9) \cup 8(9)$ under l and r are $8(36) \cup 128(144)$ and $11(18) \cup 17(18) \cup 2(36) \cup 26(36) \cup 20(72) \cup 56(144)$, respectively, thus form a partition of $2(9) \cup 8(9)$. Hence there is a mapping d such that $ld = rd = 1$. This mapping is given by

$$d :\; n \mapsto \begin{cases} \frac{n}{4} & \text{if } n \in 8(36), \\ \frac{n}{16} & \text{if } n \in 128(144), \\ \frac{3n+1}{2} & \text{if } n \in 11(18) \cup 17(18), \\ \frac{3n+2}{4} & \text{if } n \in 2(36) \cup 26(36), \\ \frac{3n+4}{8} & \text{if } n \in 20(72), \\ \frac{3n+8}{16} & \text{if } n \in 56(144), \\ n & \text{if } n \in \mathbb{Z} \setminus (2(9) \cup 8(9)). \end{cases}$$

Applying d to a vertex means going down one level in the tree.

It is not nice that the vertices of our tree $\mathcal{T}_1$ are all in $2(9) \cup 8(9)$. It would be much nicer to reformulate the problem in such a way that the question is whether a given tree with root 1 has *any* positive integer as a vertex.

We can do this as follows: First we choose an injective mapping f whose image is $2(9) \cup 8(9)$, which maps 1 to 8 and which does not map positive integers to negative ones or vice versa:

$$f :\; n \mapsto \begin{cases} \frac{9n+4}{2} & \text{if } n \in 0(2), \\ \frac{9n+7}{2} & \text{if } n \in 1(2). \end{cases}$$

Then we determine the unique mappings L, R and D such that $fl = Lf$, $fr = Rf$ and $fd = Df$, that L and R are injective and that we have $LD = RD = 1$. Straightforward calculations with the GAP [4] package RCWA [1] yield

$$L : n \mapsto \begin{cases} 4n+1 & \text{if } n \in 0(2), \\ 16n+12 & \text{if } n \in 1(2), \end{cases}$$

$$R : n \mapsto \begin{cases} \frac{4n}{3} & \text{if } n \in 0(6), \\ \frac{8n+4}{3} & \text{if } n \in 1(6), \\ \frac{16n+4}{3} & \text{if } n \in 2(6), \\ \frac{2n}{3} & \text{if } n \in 3(6), \\ \frac{4n-1}{3} & \text{if } n \in 4(6), \\ \frac{2n-1}{3} & \text{if } n \in 5(6), \end{cases} \quad \text{and}$$

$$D : n \mapsto \begin{cases} \frac{n-1}{4} & \text{if } n \in 1(8), \\ \frac{n-12}{16} & \text{if } n \in 28(32), \\ \frac{3n}{4} & \text{if } n \in 0(8), \\ \frac{3n-4}{8} & \text{if } n \in 4(16), \\ \frac{3n-4}{16} & \text{if } n \in 12(32), \\ \frac{3n}{2} & \text{if } n \in 2(4), \\ \frac{3n+1}{4} & \text{if } n \in 5(8), \\ \frac{3n+1}{2} & \text{if } n \in 3(4). \end{cases}$$

The images of L and R are $1(8) \cup 28(32)$ and $\mathbb{Z} \setminus (1(8) \cup 28(32))$, respectively. Hence they form a partition of $\mathbb{Z}$. Let $\mathcal{T}_2$ be the complete infinite binary tree with root 1 and in which the left or right child of a vertex n is $L(n)$ or $R(n)$, respectively. It is easy to see that this tree has the desired properties.

4. THE 3N+1 CONJECTURE AS AN ASSERTION ON A MONOID

In the last section, we have transformed the $3n+1$ Conjecture into the assertion that all positive integers occur as vertices of a certain tree $\mathcal{T}_2$. We can also rewrite it in terms of the following monoid:

Definition 4.1. Let $\mathcal{C}$ be the free monoid of rank 2 which is generated by the restrictions of the mappings L and R introduced in Section 3 to the positive integers.

Theorem 4.2. The $3n+1$ Conjecture is equivalent to the assertion that the mapping

$$\gamma : \mathcal{C} \rightarrow \mathbb{N}, \; c \mapsto c(1)$$

is bijective.

Remark 4.3. It is obvious that the mapping γ in Theorem 4.2 is injective. The problem is to show that it is surjective as well.

It is also easy to see that we can rewrite the $3n+1$ Conjecture in the following way as a conjugacy problem for monoids:

Theorem 4.4. The $3n+1$ Conjecture is equivalent to the assertion that the monoids $\mathcal{C}$ and $\mathcal{C}_0 := \langle n \mapsto 2n, n \mapsto 2n+1 \rangle$ are conjugate in the full symmetric group $\mathrm{Sym}(\mathbb{N})$.

REFERENCES

1. Stefan Kohl, *RCWA - Residue-Class-Wise Affine Groups; Version 2.4.3*, 2007, GAP package, published at http://www.gap-system.org/Packages/rcwa.html.
2. Jeffrey C. Lagarias, *The 3x+1 problem and its generalizations*, Amer. Math. Monthly **92** (1985), 1–23.
3. ______, *The 3x+1 problem: An annotated bibliography*, 2006, http://arxiv.org/abs/math.NT/0309224.
4. The GAP Group, *GAP – Groups, Algorithms, and Programming; Version 4.4.9*, 2006, http://www.gap-system.org.

INSTITUT FÜR GEOMETRIE UND TOPOLOGIE, PFAFFENWALDRING 57, UNIVERSITÄT STUTTGART 70550 STUTTGART, GERMANY

E-mail address: kohl@mathematik.uni-stuttgart.de