ON THE CYCLE STRUCTURE OF PRODUCTS OF TWO CLASS TRANSPOSITIONS

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ABSTRACT. We multiply two permutations interchanging disjoint residue classes of $\mathbb{Z}$ which are affine on each residue class and which fix $\mathbb{N}_0$ setwise, and ask for order and cycle structure of the product. By means of computation we find 88 different cases occurring for products of such permutations interchanging residue classes with moduli $\leqslant 6$.

1. INTRODUCTION

In this note we investigate products of two permutations of $\mathbb{Z}$ of the following type:

Definition 1.1. By $r(m)$ we denote the residue class $r + m\mathbb{Z}$. Given two disjoint residue classes $r_1(m_1)$ and $r_2(m_2)$, let the *class transposition* $(r_1(m_1), r_2(m_2)) \in \mathrm{Sym}(\mathbb{Z})$ be the permutation which interchanges $r_1 + km_1$ and $r_2 + km_2$ for each integer k and which fixes all other points. Here we assume that $0 \leqslant r_1 < m_1$ and that $0 \leqslant r_2 < m_2$.

The set of all such permutations of $\mathbb{Z}$ generates the countable simple group $\mathrm{CT}(\mathbb{Z})$ discussed in [3].

The properties of a product of two class transpositions depend on how the 4 involved residue classes intersect each other. We exhibit 18 essentially different possibilities for this. However these intersection types are still far from fully determining the cycle structure of the product. – Even restricting our attention to residue classes $r_i(m_i)$ with moduli $m_i \leqslant 6$, by means of computation we exhibit a total of 88 different subcases.

2. THE 18 INTERSECTION TYPES

We want to describe the products of two class transpositions 'as good as possible'. In particular we are interested in their cycle structure.

Let $(r_1(m_1), r_2(m_2))$ and $(r_3(m_3), r_4(m_4))$ be two class transpositions. Obviously it matters in which ways the residue classes $r_1(m_1), \dots, r_4(m_4)$ intersect each other. By definition we have $r_1(m_1) \cap r_2(m_2) = r_3(m_3) \cap r_4(m_4) = \emptyset$, hence we can restrict our attention to the following 4 pairs of residue classes:

$$\{r_1(m_1), r_3(m_3)\},\ \{r_1(m_1), r_4(m_4)\},\ \{r_2(m_2), r_3(m_3)\},\ \{r_2(m_2), r_4(m_4)\}.$$

For purposes of notation, we number these pairs from 1 to 4 in the order in which they are listed. Let $(r_i(m_i), r_j(m_j))$ be one of them. We need to distinguish the following cases:

(1) $r_i(m_i) = r_j(m_j)$, in the sequel denoted by 0,
(2) $r_i(m_i) \supset r_j(m_j)$, in the sequel denoted by 1,
(3) $r_i(m_i) \subset r_j(m_j)$, in the sequel denoted by 2,
(4) $r_i(m_i) \cap r_j(m_j) = \emptyset$, in the sequel denoted by 3, and
(5) $r_i(m_i) \cap r_j(m_j) = r_k(m_k)$ where $r_k(m_k)$ is neither $r_i(m_i)$ nor $r_j(m_j)$, in the sequel denoted by 4.

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We write down the 'intersection type' of a pair of class transpositions as a 4-tuple $t = (t_1, \dots, t_4)$ of integers from 0 to 4, where t_i describes how the residue classes in the i-th pair intersect each other. So for example the intersection type $(1, 4, 3, 2)$ means that $r_1(m_1) \supset r_3(m_3)$, that $r_1(m_1) \cap r_4(m_4) = r_k(m_k)$, where $r_k(m_k)$ is a residue class other than $r_1(m_1)$ and $r_4(m_4)$, that $r_2(m_2) \cap r_3(m_3) = \emptyset$ and that $r_2(m_2) \subset r_4(m_4)$.

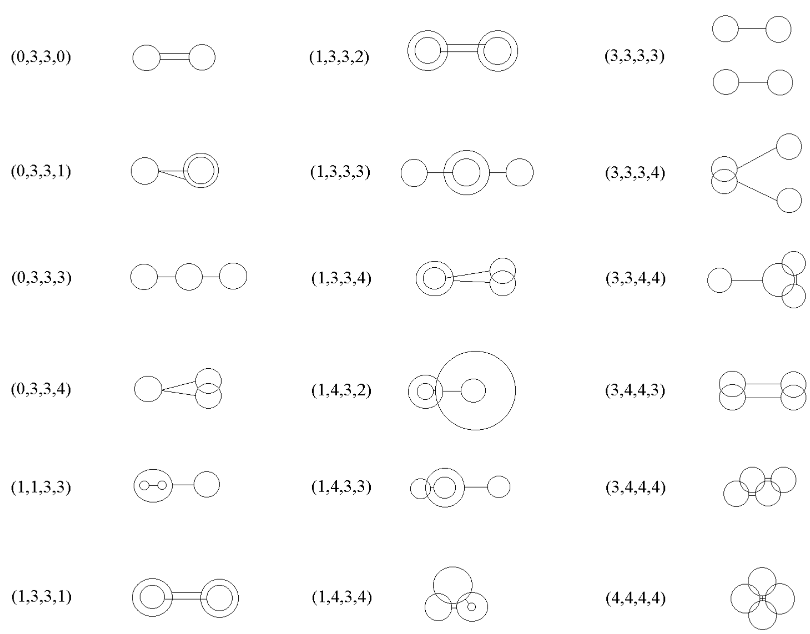
In total, there are $5^4 = 625$ tuples describing conceivable intersection types. We need to determine which of them are really possible and essentially different.

The Klein 4-group $\mathrm{V}_4 = \langle (1,2)(3,4), (1,3)(2,4) \rangle$ acts on the set of our 4-tuples by permuting the entries. Tuples lying in the same orbit under this action describe the same intersection type, thus we can restrict our attention to orbit representatives. This still leaves us 175 cases. In the sequel, we assume that we have chosen the lexicographically smallest orbit representatives.

We observe that many of the remaining intersection types are contradictory, and that some others are equivalent. Let $t = (t_1, t_2, t_3, t_4)$ denote an intersection type. Then:

(1) None of the pairs (t_1, t_2), (t_3, t_4), (t_1, t_3), (t_2, t_4) can be $(0, 0)$, $(1, 2)$ or $(2, 1)$. This rules out 64 cases, and leaves us 111.
(2) If $2 \in t$ but not $1 \in t$, we get an equivalent intersection type if we replace any 2 in t by 1. This rules out further 42 cases, and leaves us 69.
(3) At most two entries of t can be equal to 1, and if indeed two are equal to 1, there cannot be a 0 or a 4 in t. This rules out further 17 cases, and leaves us 52.
(4) We have $(t_1, t_2) \in \{(0, 3), (1, 1), (1, 3), (1, 4), (3, 3), (3, 4), (4, 4)\}$, and (t_1, t_3) is not $(0, 1)$, $(1, 1)$ or $(1, 4)$. This rules out further 29 cases, and leaves us 23.
(5) If $t_1 = 0$, then $4 \notin \{t_2, t_3\}$. Also, $(3, 4, 3, 4)$ is equivalent to $(3, 3, 4, 4)$. This rules out further 5 cases, and finally leaves us 18.

Visualizing the pairs of residue classes which are interchanged by the class transpositions $(r_1(m_1), r_2(m_2))$ respectively $(r_3(m_3), r_4(m_4))$ as pairs of circles connected by lines, the remaining 18 essentially different intersection types can be depicted as follows:



3. Computational Results

In this section we determine by means of computation what can happen in each of the 18 situations worked out in the previous section when the moduli of the residue classes $r_1(m_1), \dots, r_4(m_4)$ are all $\leqslant 6$. This is done with the GAP [1] package RCWA [4]. For a description of some of the algorithms implemented in this package, see [2].

There are $\sum_{m=1}^{6} m = 21$ residue classes with modulus $\leqslant 6$. Among the $\binom{21}{2} = 210$ unordered pairs of distinct such residue classes, 69 are disjoint:

$$\{0(2),1(2)\}, \{0(2),1(4)\}, \{0(2),1(6)\}, \{0(2),3(4)\}, \{0(2),3(6)\}, \{0(2),5(6)\},$$
$$\{0(3),1(3)\}, \{0(3),1(6)\}, \{0(3),2(3)\}, \{0(3),2(6)\}, \{0(3),4(6)\}, \{0(3),5(6)\},$$
$$\{0(4),1(4)\}, \{0(4),1(6)\}, \{0(4),2(4)\}, \{0(4),3(4)\}, \{0(4),3(6)\}, \{0(4),5(6)\},$$
$$\{0(5),1(5)\}, \{0(5),2(5)\}, \{0(5),3(5)\}, \{0(5),4(5)\}, \{0(6),1(6)\}, \{0(6),2(6)\},$$
$$\{0(6),3(6)\}, \{0(6),4(6)\}, \{0(6),5(6)\}, \{1(2),0(4)\}, \{1(2),0(6)\}, \{1(2),2(4)\},$$
$$\{1(2),2(6)\}, \{1(2),4(6)\}, \{1(3),0(6)\}, \{1(3),2(3)\}, \{1(3),2(6)\}, \{1(3),3(6)\},$$
$$\{1(3),5(6)\}, \{1(4),0(6)\}, \{1(4),2(4)\}, \{1(4),2(6)\}, \{1(4),3(4)\}, \{1(4),4(6)\},$$
$$\{1(5),2(5)\}, \{1(5),3(5)\}, \{1(5),4(5)\}, \{1(6),2(6)\}, \{1(6),3(6)\}, \{1(6),4(6)\},$$
$$\{1(6),5(6)\}, \{2(3),0(6)\}, \{2(3),1(6)\}, \{2(3),3(6)\}, \{2(3),4(6)\}, \{2(4),1(6)\},$$
$$\{2(4),3(4)\}, \{2(4),3(6)\}, \{2(4),5(6)\}, \{2(5),3(5)\}, \{2(5),4(5)\}, \{2(6),3(6)\},$$
$$\{2(6),4(6)\}, \{2(6),5(6)\}, \{3(4),0(6)\}, \{3(4),2(6)\}, \{3(4),4(6)\}, \{3(5),4(5)\},$$
$$\{3(6),4(6)\}, \{3(6),5(6)\}, \{4(6),5(6)\}.$$

Thus there are $\binom{70}{2} = 2415$ unordered pairs of class transpositions which interchange such residue classes. Investigating the cycle structure of their products, we find 88 different cases. It seems not unlikely that still more cases occur if one increases the bound on the moduli of the residue classes. From now on, σ always denotes the product of a pair of class transpositions. In the table below, for each of the 88 cases we give the following data:

(1) The intersection type, as described above.
(2) The order of σ.
(3) The *cycle type* of σ. This is a pair consisting of the set of cycle lengths of σ, where the set-theoretic union of the finite cycles of a given length is always a union of finitely many residue classes, and the number of fixed points of σ stemming from non-identity affine partial mappings.
(4) The number of pairs which correspond to the particular case.
(5) An example.

In the examples, we write down the class transpositions and the interchanged residue classes in lexicographic order.

Inter.-Type	ord(σ)	Cycle Type of σ	# of Pairs	Example
(0,3,3,0)	1	$(\{1\},0)$	69	$(0(2),1(2)) \cdot (0(2),1(2))$
(0,3,3,1)	∞	$(\{\infty\},0)$	2	$(0(2),1(2)) \cdot (1(2),2(6))$
	∞	$(\{\infty\},2)$	8	$(0(2),1(2)) \cdot (0(2),1(4))$
	∞	$(\{1,\infty\},0)$	10	$(0(2),1(6)) \cdot (1(6),2(6))$
	∞	$(\{1,\infty\},2)$	76	$(0(2),1(4)) \cdot (0(4),1(4))$
(0,3,3,3)	3	$(\{3\},0)$	9	$(0(3),1(3)) \cdot (0(3),2(3))$
	3	$(\{1,3\},0)$	246	$(0(4),1(4)) \cdot (1(4),2(4))$
(0,3,3,4)	∞	$(1(2),2)$	108	$(0(2),1(4)) \cdot (0(2),1(6))$
				To be continued.

Continued.

Inter.-Type	$\mathrm{ord}(\sigma)$	Cycle Type of σ	# of Pairs	Example
(1,1,3,3)	4	$(\{4\},0)$	2	$(0(2),1(2)) \cdot (0(4),2(4))$
	4	$(\{1,4\},0)$	28	$(0(2),1(4)) \cdot (0(4),2(4))$
	4	$(\{2,4\},0)$	6	$(0(2),1(2)) \cdot (0(6),2(6))$
	4	$(\{1,2,4\},0)$	30	$(0(2),1(6)) \cdot (0(6),2(6))$
(1,3,3,1)	2	$(\{1,2\},0)$	11	$(0(2),1(2)) \cdot (0(4),1(4))$
	4	$(\{4\},0)$	2	$(0(2),1(2)) \cdot (1(4),2(4))$
	4	$(\{1,4\},0)$	6	$(0(3),1(3)) \cdot (1(6),3(6))$
	4	$(\{2,4\},0)$	6	$(0(2),1(2)) \cdot (1(6),2(6))$
	∞	$(0(2),2)$	12	$(0(2),1(2)) \cdot (0(4),1(6))$
(1,3,3,2)	∞	$(\{\infty\},0)$	17	$(0(2),1(4)) \cdot (1(2),2(4))$
	∞	$(\{\infty\},2)$	8	$(0(2),1(4)) \cdot (1(2),0(4))$
	∞	$(\{1,\infty\},0)$	6	$(0(3),1(6)) \cdot (1(3),3(6))$
	∞	$(\{1,\infty\},2)$	6	$(0(3),1(6)) \cdot (1(3),0(6))$
(1,3,3,3)	6	$(\{2,3\},0)$	32	$(1(2),0(4)) \cdot (1(4),2(4))$
	6	$(\{1,2,3\},0)$	180	$(1(2),2(6)) \cdot (0(6),1(6))$
(1,3,3,4)	12	$(\{1,3,4\},0)$	12	$(0(2),1(6)) \cdot (1(4),2(4))$
	12	$(\{1,2,3,4\},0)$	24	$(0(2),1(4)) \cdot (1(6),2(6))$
	∞	$(\{1,2,\infty\},0)$	4	$(0(2),1(6)) \cdot (2(4),3(4))$
	∞	$(\{1,2,\infty\},2)$	20	$(0(2),1(4)) \cdot (0(6),1(6))$
	∞	$(\mathbb{N},0)$	28	$(0(2),1(6)) \cdot (1(4),2(6))$
	∞	$(\mathbb{N},2)$	32	$(0(2),1(4)) \cdot (0(4),1(6))$
(1,4,3,2)	∞	$(\{1,\infty\},0)$	6	$(0(2),1(6)) \cdot (1(3),2(6))$
	∞	$(\{1,\infty\},2)$	6	$(0(2),1(6)) \cdot (1(3),0(6))$
(1,4,3,3)	12	$(\{1,3,4\},0)$	12	$(1(2),0(6)) \cdot (0(3),2(6))$
	12	$(\{1,2,3,4\},0)$	48	$(0(2),1(6)) \cdot (0(3),2(6))$
(1,4,3,4)	4	$(\{4\},0)$	6	$(0(2),1(2)) \cdot (1(3),2(6))$
	6	$(\{1,3,6\},0)$	4	$(1(2),0(6)) \cdot (0(3),2(3))$
	6	$(\{1,2,3,6\},0)$	8	$(0(2),1(4)) \cdot (2(3),0(6))$
	12	$(\{1,3,4\},0)$	16	$(0(3),2(3)) \cdot (1(4),0(6))$
	12	$(\{1,3,4,6\},0)$	12	$(1(3),0(6)) \cdot (2(4),1(6))$
	∞	$(\{1,\infty\},0)$	2	$(0(2),1(6)) \cdot (1(3),2(3))$
	∞	$(\{1,\infty\},2)$	6	$(0(2),1(6)) \cdot (0(3),1(3))$
	∞	$(\{2,\infty\},2)$	6	$(0(2),1(2)) \cdot (0(3),1(6))$
	∞	$(\{1,2,\infty\},0)$	4	$(0(3),5(6)) \cdot (1(4),0(6))$
	∞	$(\{1,2,\infty\},2)$	16	$(0(2),1(4)) \cdot (0(3),4(6))$
	∞	$(\mathbb{N},2)$	16	$(0(3),1(3)) \cdot (0(4),1(6))$
(3,3,3,3)	2	$(\{2\},0)$	18	$(0(4),2(4)) \cdot (1(4),3(4))$
	2	$(\{1,2\},0)$	126	$(0(5),1(5)) \cdot (2(5),3(5))$
(3,3,3,4)	6	$(\{2,3\},0)$	12	$(1(2),0(6)) \cdot (1(3),2(6))$
	6	$(\{1,2,3\},0)$	120	$(1(4),0(6)) \cdot (1(6),2(6))$
(3,3,4,4)	4	$(\{2,4\},0)$	2	$(0(2),1(6)) \cdot (0(3),2(3))$
	4	$(\{1,2,4\},0)$	24	$(2(4),1(6)) \cdot (0(6),4(6))$
	6	$(\{2,3\},0)$	10	$(1(2),0(6)) \cdot (1(3),2(3))$
	6	$(\{1,2,3\},0)$	126	$(0(4),1(6)) \cdot (0(6),2(6))$
	12	$(\{1,2,3,4\},0)$	24	$(2(3),0(6)) \cdot (0(4),1(6))$

To be continued.

Continued.

Inter.-Type	$\mathrm{ord}(\sigma)$	Cycle Type of σ	# of Pairs	Example
(3,4,4,3)	2	$(\{1,2\},0)$	10	$(0(4),1(4)) \cdot (0(6),1(6))$
	4	$(\{1,2,4\},0)$	10	$(0(4),3(4)) \cdot (1(6),2(6))$
	6	$(\{1,2,3\},0)$	16	$(1(4),2(4)) \cdot (0(6),1(6))$
	∞	$(\mathbb{N},0)$	26	$(0(4),1(6)) \cdot (1(4),2(6))$
	∞	$(\mathbb{N},2)$	10	$(0(4),1(6)) \cdot (1(4),0(6))$
(3,4,4,4)	6	$(\{1,2,3\},0)$	6	$(0(2),1(4)) \cdot (0(3),1(6))$
	10	$(\{1,2,5\},0)$	2	$(0(2),1(4)) \cdot (2(3),1(6))$
	12	$(\{1,3,4\},0)$	4	$(1(2),0(4)) \cdot (1(3),2(6))$
	12	$(\{1,2,3,4\},0)$	22	$(0(3),1(6)) \cdot (1(4),2(4))$
	20	$(\{1,2,4,5\},0)$	8	$(0(3),1(6)) \cdot (2(4),3(4))$
	30	$(\{1,2,3,5\},0)$	18	$(1(3),2(6)) \cdot (0(4),3(4))$
	∞	$(\{1,2,3,\infty\},2)$	12	$(0(3),1(6)) \cdot (0(4),1(4))$
(4,4,4,4)	4	$(\{1,4\},0)$	6	$(0(2),1(2)) \cdot (0(3),1(3))$
	4	$(\{2,4\},0)$	5	$(0(2),1(2)) \cdot (0(5),2(5))$
	4	$(\{1,2,4\},0)$	40	$(0(3),2(3)) \cdot (1(4),3(4))$
	6	$(\{6\},0)$	1	$(0(2),1(2)) \cdot (0(3),2(3))$
	6	$(\{1,6\},0)$	2	$(0(4),2(4)) \cdot (0(6),4(6))$
	6	$(\{2,6\},0)$	1	$(0(2),1(2)) \cdot (0(5),4(5))$
	6	$(\{1,2,3\},0)$	162	$(0(3),1(3)) \cdot (0(4),1(4))$
	6	$(\{1,2,6\},0)$	13	$(1(3),2(3)) \cdot (0(4),3(4))$
	12	$(\{1,3,4\},0)$	7	$(0(3),1(3)) \cdot (0(4),2(4))$
	12	$(\{1,2,3,4\},0)$	164	$(0(3),2(3)) \cdot (0(5),1(5))$
	15	$(\{1,3,5\},0)$	4	$(0(2),1(4)) \cdot (0(3),2(3))$
	20	$(\{1,4,5\},0)$	2	$(0(3),1(6)) \cdot (1(4),3(4))$
	30	$(\{1,2,3,5\},0)$	76	$(0(2),1(4)) \cdot (0(5),2(5))$
	30	$(\{1,3,5,6\},0)$	4	$(0(2),1(4)) \cdot (1(3),2(3))$
	30	$(\{1,2,3,5,6\},0)$	22	$(1(2),2(4)) \cdot (0(5),1(5))$
	60	$(\{1,2,3,4,5\},0)$	16	$(1(3),0(6)) \cdot (1(5),2(5))$
	60	$(\{1,2,3,4,5,6\},0)$	28	$(1(4),0(6)) \cdot (2(5),3(5))$
	∞	$(\{1,2,\infty\},4)$	4	$(0(2),1(4)) \cdot (0(3),1(3))$
	∞	$(\{1,3,\infty\},2)$	10	$(1(3),0(6)) \cdot (0(4),2(4))$
	∞	$(\{1,2,3,\infty\},2)$	26	$(0(3),1(6)) \cdot (0(5),1(5))$
	∞	$(\{1,2,3,4,\infty\},0)$	6	$(1(2),0(6)) \cdot (2(5),3(5))$
	∞	$(\{1,2,3,4,\infty\},2)$	22	$(0(2),1(4)) \cdot (0(5),1(5))$
	∞	$(0(2) \cup \{1\},4)$	20	$(0(4),1(6)) \cdot (0(5),1(5))$
			Total: 2415	

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