A BOUND ON THE ORDER OF THE OUTER AUTOMORPHISM GROUP OF
A FINITE SIMPLE GROUP OF GIVEN ORDER

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ABSTRACT. We prove that the order of the outer automorphism group of a non-abelian
finite simple group of order $n$ is bounded by $2 \log n$, and show that this bound is sharp in
the sense that it cannot be improved by more than a small constant factor. The proof uses
the classification of finite simple groups.

Theorem 1. Let $G$ be a non-abelian finite simple group. Then we have

$|\text{Out}(G)| < 2 \log |G|$. 

Proof. We use the classification of finite simple groups (see e.g. [1], in particular p. xvi,
Table 5, 6). In the following, let $q = p^f$ denote a prime power and $n$ a positive integer.

(1) Let $G$ be an alternating group. Then we have $|\text{Out}(G)| \leq 4 < 2 \log 60 \leq 2 \log |G|.$

(2) Let $G$ be either one of the 26 sporadic simple groups or the Tits group. Then we
have $|\text{Out}(G)| \leq 2 < 2 \log 7920 \leq 2 \log |G|.$

(3) Assume $G \cong A_1(q) \cong PSL(2, q)$. Then we have

$|\text{Out}(G)| \leq 2f \leq 2 \cdot 2 \log q = 2 \log(q^2)$

$< 2 \log \left( \frac{1}{2} q(q^2 - 1) \right) \leq 2 \log |G|.$

(4) Assume $G \cong A_2(q) \cong PSL(3, q)$. Then we have

$|\text{Out}(G)| \leq 6f \leq 6 \cdot 2 \log q = 2 \log(q^6)$

$< 2 \log \left( \frac{1}{(3, q - 1)} q^3(q^2 - 1)(q^3 - 1) \right) = 2 \log |G|.$

(5) Assume $G \cong 2A_2(q) \cong PSU(3, q)$. Then we have

$|\text{Out}(G)| \leq 6f \leq 6 \cdot 2 \log q = 2 \log(q^6)$

$< 2 \log \left( \frac{1}{(3, q + 1)} q^3(q^2 - 1)(q^3 + 1) \right) = 2 \log |G|.$

(6) Assume $G \cong A_3(q) \cong PSL(4, q)$ or $G \cong 2A_3(q) \cong PSU(4, q)$. Then we have

$|\text{Out}(G)| \leq 8f \leq 8 \cdot 2 \log q = 2 \log(q^8)$

$< 2 \log \left( \frac{1}{4} q^6(q^2 - 1)(q^3 - 1)(q^4 - 1) \right) \leq 2 \log |G|.$

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(7) Assume $G \cong A_n(q) \cong PSL(n+1,q)$ or $G \cong 2A_n(q) \cong PSU(n+1,q)$, where $n \geq 4$. Then we have

$$|\text{Out}(G)| \leq 2(n+1)f \leq 2(n+1) \cdot 2 \log q$$

$$\leq \frac{n(n+1)}{2} \cdot 2 \log q = 2 \log \left( q^{\frac{n(n+1)}{2}} \right)$$

$$\leq 2 \log \left( \frac{1}{q+1} q^{\frac{n(n+1)}{2}} (q^2 - 1) \right) < 2 \log |G|.$$  

(8) Assume $G \cong B_n(q) \cong O(2n+1,q)$. Then we have

$$|\text{Out}(G)| \leq 2f \leq 2 \cdot 2 \log q$$

$$< n^2 \cdot 2 \log q = 2 \log \left( q^{n^2} \right) < 2 \log |G|.$$  

(9) Assume $G \cong 2B_2(q) \cong Sz(q)$. Then we have $|\text{Out}(G)| = 2 \log q < 2 \log |G|$.

(10) Assume $G \cong C_n(q) \cong PSp(2n,q)$. Then we have

$$|\text{Out}(G)| \leq 2f \leq 2 \cdot 2 \log q$$

$$< n^2 \cdot 2 \log q = 2 \log \left( q^{n^2} \right) < 2 \log |G|.$$  

(11) Assume $G \cong D_4(q) \cong O^+(8,q)$. Then we have

$$|\text{Out}(G)| \leq 24f \leq 24 \cdot 2 \log q = 2 \log(q^{24})$$

$$< 2 \log \left( \frac{1}{q, q^4 - 1} q^{12(q^4 - 1)(q^2 - 1)(q^4 - 1)(q^6 - 1)} \right)$$

$$= 2 \log |G|.$$  

(12) Assume $G \cong D_n(q) \cong O^+(2n,q)$ where $n \geq 5$, or $G \cong 2D_n(q) \cong O^-(2n,q)$ where $n \geq 4$. Then we have

$$|\text{Out}(G)| \leq 8f \leq 8 \cdot 2 \log q$$

$$< n^2 \cdot 2 \log q = 2 \log \left( q^{n^2} \right) < 2 \log |G|.$$  

(13) Assume $G \in \{3D_4(q), G_2(q), 2G_2(q), F_4(q), 2F_4(q), E_6(q), 2E_6(q), E_7(q), E_8(q)\}$. Then we have $|\text{Out}(G)| \leq 6f \leq 6 \cdot 2 \log q = 2 \log(q^6) < 2 \log |G|$. \hfill $\square$

Remark 2. The bound in Theorem 1 cannot be improved significantly. If $f$ is even and $q = 2^f$, we have

$$|\text{Out}(A_2(q))| = \frac{6f}{2 \log |A_2(q)|} \xrightarrow{f \to \infty} \frac{3}{4}.$$

Hence we even cannot replace our bound by $\ln |G|$ – even not if we allow a finite number of exceptions.

REFERENCES