A BOUND ON THE ORDER OF THE OUTER AUTOMORPHISM GROUP OF A FINITE SIMPLE GROUP OF GIVEN ORDER

STEFAN KOHL

ABSTRACT. We prove that the order of the outer automorphism group of a non-abelian finite simple group of order n is bounded by ${}_2\log n$, and show that this bound is sharp in the sense that it cannot be improved by more than a small constant factor. The proof uses the classification of finite simple groups.

Theorem 1. *Let G be a non-abelian finite simple group. Then we have*

$$|\mathrm{Out}(G)| \;<\; {}_2\log|G|.$$

Proof. We use the classification of finite simple groups (see e.g. [1], in particular p. xvi, Table 5, 6). In the following, let $q = p^f$ denote a prime power and n a positive integer.

(1) Let G be an alternating group. Then we have $|\mathrm{Out}(G)| \le 4 < {}_2\log 60 \le {}_2\log|G|$.

(2) Let G be either one of the 26 sporadic simple groups or the Tits group. Then we have $|\mathrm{Out}(G)| \le 2 < {}_2\log 7920 \le {}_2\log|G|$.

(3) Assume $G \cong \mathrm{A}_1(q) \cong \mathrm{PSL}(2,q)$. Then we have

$$|\mathrm{Out}(G)| \le 2f \le 2\cdot {}_2\log q = {}_2\log(q^2) \overset{q\geqslant 4}{<} {}_2\log\left(\frac{1}{2}\,q(q^2-1)\right) \le {}_2\log|G|.$$

(4) Assume $G \cong \mathrm{A}_2(q) \cong \mathrm{PSL}(3,q)$. Then we have

$$|\mathrm{Out}(G)| \le 6f \le 6\cdot {}_2\log q = {}_2\log(q^6) < {}_2\log\left(\frac{1}{(3,q-1)}\,q^3(q^2-1)(q^3-1)\right) = {}_2\log|G|.$$

(5) Assume $G \cong {}^2\mathrm{A}_2(q) \cong \mathrm{PSU}(3,q)$. Then we have

$$|\mathrm{Out}(G)| \le 6f \le 6\cdot {}_2\log q = {}_2\log(q^6) < {}_2\log\left(\frac{1}{(3,q+1)}\,q^3(q^2-1)(q^3+1)\right) = {}_2\log|G|.$$

(6) Assume $G \cong \mathrm{A}_3(q) \cong \mathrm{PSL}(4,q)$ or $G \cong {}^2\mathrm{A}_3(q) \cong \mathrm{PSU}(4,q)$. Then we have

$$|\mathrm{Out}(G)| \le 8f \le 8\cdot {}_2\log q = {}_2\log(q^8) < {}_2\log\left(\frac{1}{4}\,q^6(q^2-1)(q^3-1)(q^4-1)\right) \le {}_2\log|G|.$$

2000 *Mathematics Subject Classification.* Primary 20E32, 20D06, 20D45.

(7) Assume $G \cong \mathrm{A}_n(q) \cong \mathrm{PSL}(n+1,q)$ or $G \cong {}^2\mathrm{A}_n(q) \cong \mathrm{PSU}(n+1,q)$, where $n \geq 4$. Then we have

$$\begin{aligned} |\mathrm{Out}(G)| &\leq 2(n+1)f \leq \ 2(n+1) \cdot {}_2\log q \\ &\leq \frac{n(n+1)}{2} \cdot {}_2\log q \ = \ {}_2\log\left(q^{\frac{n(n+1)}{2}}\right) \\ &\leq {}_2\log\left(\frac{1}{q+1}\, q^{\frac{n(n+1)}{2}}(q^2-1)\right) \ < \ {}_2\log|G|. \end{aligned}$$

(8) Assume $G \cong \mathrm{B}_n(q) \cong \mathrm{O}(2n+1,q)$. Then we have

$$\begin{aligned} |\mathrm{Out}(G)| &\leq 2f \ \leq \ 2 \cdot {}_2\log q \\ &< n^2 \cdot {}_2\log q \ = \ {}_2\log\left(q^{n^2}\right) \ < \ {}_2\log|G|. \end{aligned}$$

(9) Assume $G \cong {}^2\mathrm{B}_2(q) \cong \mathrm{Sz}(q)$. Then we have $|\mathrm{Out}(G)| = {}_2\log q < {}_2\log|G|$.

(10) Assume $G \cong \mathrm{C}_n(q) \cong \mathrm{PSp}(2n,q)$. Then we have

$$\begin{aligned} |\mathrm{Out}(G)| &\leq 2f \leq \ 2 \cdot {}_2\log q \\ &< n^2 \cdot {}_2\log q \ = \ {}_2\log\left(q^{n^2}\right) \ < \ {}_2\log|G|. \end{aligned}$$

(11) Assume $G \cong \mathrm{D}_4(q) \cong \mathrm{O}^+(8,q)$. Then we have

$$\begin{aligned} |\mathrm{Out}(G)| &\leq 24f \leq \ 24 \cdot {}_2\log q \ = \ {}_2\log(q^{24}) \\ &< {}_2\log\left(\frac{1}{(4,q^4-1)}\, q^{12}(q^4-1)(q^2-1)(q^4-1)(q^6-1)\right) \\ &= {}_2\log|G|. \end{aligned}$$

(12) Assume $G \cong \mathrm{D}_n(q) \cong \mathrm{O}^+(2n,q)$ where $n \geq 5$, or $G \cong {}^2\mathrm{D}_n(q) \cong \mathrm{O}^-(2n,q)$ where $n \geq 4$. Then we have

$$\begin{aligned} |\mathrm{Out}(G)| &\leq 8f \leq \ 8 \cdot {}_2\log q \\ &\overset{n\geqslant 4}{<} \ n^2 \cdot {}_2\log q \ = \ {}_2\log\left(q^{n^2}\right) \ < \ {}_2\log|G|. \end{aligned}$$

(13) Assume $G \in \{{}^3\mathrm{D}_4(q), \mathrm{G}_2(q), {}^2\mathrm{G}_2(q), \mathrm{F}_4(q), {}^2\mathrm{F}_4(q), \mathrm{E}_6(q), {}^2\mathrm{E}_6(q), \mathrm{E}_7(q), \mathrm{E}_8(q)\}$. Then we have $|\mathrm{Out}(G)| \leq 6f \leq 6 \cdot {}_2\log q = {}_2\log(q^6) < {}_2\log|G|$. □

Remark 2. The bound in Theorem 1 cannot be improved significantly. If f is even and $q = 2^f$, we have

$$\frac{|\mathrm{Out}(\mathrm{A}_2(q))|}{{}_2\log|\mathrm{A}_2(q)|} \ = \ \frac{6f}{{}_2\log\left(\frac{1}{3}\, q^3(q^2-1)(q^3-1)\right)} \ \underset{f\to\infty}{\longrightarrow} \ \frac{3}{4}.$$

Hence we even cannot replace our bound by $\ln|G|$ – even not if we allow a finite number of exceptions.

References

1. John H. Conway, Robert T. Curtis, Simon P. Norton, Richard A. Parker, and Robert A. Wilson, *Atlas of finite groups*, Oxford University Press, 1985.

Institut für Geometrie und Topologie, Pfaffenwaldring 57, Universität Stuttgart 70550 Stuttgart, Germany

E-mail address: kohl@mathematik.uni-stuttgart.de