Theorem *Let $n = p_1p_2p_3$ be a Carmichael number with 3 prime factors. Without loss of generality, assume that $p_2 < p_3$. Then the following hold:*

1. $p_2 < 2p_1^2$.
2. $p_3 < 2p_1^3$.
3. $n < 4p_1^6$.

Proof: It is an elementary property of Carmichael numbers $n = p_1 \cdot \ldots \cdot p_k$ that $\forall i \in \{1, \ldots, k\}$ $(p_i - 1)|(n-1)$. From this, in case $k = 3$ we conclude

- $(p_3 - 1)|(p_1p_2 - 1) \Rightarrow p_1p_2 - 1 = a(p_3 - 1)$ for some $a \in \mathbb{N}$, and
- $(p_2 - 1)|(p_1p_3 - 1) \Rightarrow p_1p_3 - 1 = b(p_2 - 1)$ for some $b \in \mathbb{N}$.

The assumption $p_2 < p_3$ implies $a < b$. Further it is

- $a \geq 2$ since $p_1p_2 - 1 = 1(p_3 - 1)$ contradicts the primality of p_3, and
- $b \geq 2$ since $p_1p_3 - 1 = 1(p_2 - 1)$ contradicts the primality of p_2.

We also get

- $p_1p_2 - 1 = a(p_3 - 1) = ap_3 - a \Longrightarrow p_3 = (p_1p_2 + a - 1)/a$, and
- $p_1p_3 - 1 = b(p_2 - 1) = bp_2 - b \Longrightarrow p_2 = (p_1p_3 + b - 1)/b$.

Inserting the former equation into the latter yields

$$p_2 = \frac{p_1(p_1p_2 + a - 1)/a + b - 1}{b} = \frac{p_1^2p_2 + ab + (p_1 - 1)a - p_1}{ab}.$$

This implies that

$$(p_1^2 - ab)p_2 + ab + (p_1 - 1)a - p_1 = 0,$$

from which we get

$$p_2 = \frac{ab + (p_1 - 1)a - p_1}{ab - p_1^2} < \frac{ab}{ab - p_1^2} + \frac{(p_1 - 1)a}{ab - p_1^2}.$$

We have $ab \neq p_1^2$, since assuming the contrary yields $a = b = p_1 \Rightarrow p_1p_2 - 1 = p_1(p_3 - 1) \Rightarrow p_3 = p_2 + 1 - 1/p_1 \notin \mathbb{N}$, which is not possible. Further it is $ab > p_1^2$ since $p_2 > 0$ and $ab + (p_1 - 1)a > 0$. This yields

$$\frac{ab}{ab - p_1^2} \leq \frac{p_1^2 + 1}{(p_1^2 + 1) - p_1^2} = p_1^2 + 1.$$

By the assumption $p_2 < p_3$ we have $p_1(p_3 - 1) > p_1p_2 - 1 = a(p_3 - 1)$, hence $a < p_1$. We conclude that

$$\frac{(p_1 - 1)a}{ab - p_1^2} < \frac{(p_1 - 1)p_1}{ab - p_1^2} \leq p_1^2 - p_1.$$

This yields $p_2 < p_1^2 + 1 + p_1^2 - p_1 < 2p_1^2$, as claimed. From $p_1p_2 - 1 = a(p_3 - 1)$ we get $p_3 < p_1p_2 < 2p_1^3$, and finally $n = p_1p_2p_3 < p_1 \cdot 2p_1^2 \cdot 2p_1^3 = 4p_1^6$. □