The Group Elements

Definition: Let R be an infinite Euclidean ring all of whose residue class rings are finite, e.g.

$$R = \mathbb{Z},\ \mathbb{Z}[i],\ \mathbb{F}_q[x],\ \mathbb{Z}_{(\pi)},\ \dots .$$

We call a mapping $f : R \to R$ *residue class-wise affine* if there is a modulus $m \in R \setminus \{0\}$ such that all restrictions $f|_{r(m)}$ of f to residue classes $r(m) \in R/mR$ are affine, i.e. of the form

$$n \longmapsto \frac{a_{r(m)} \cdot n + b_{r(m)}}{c_{r(m)}}$$

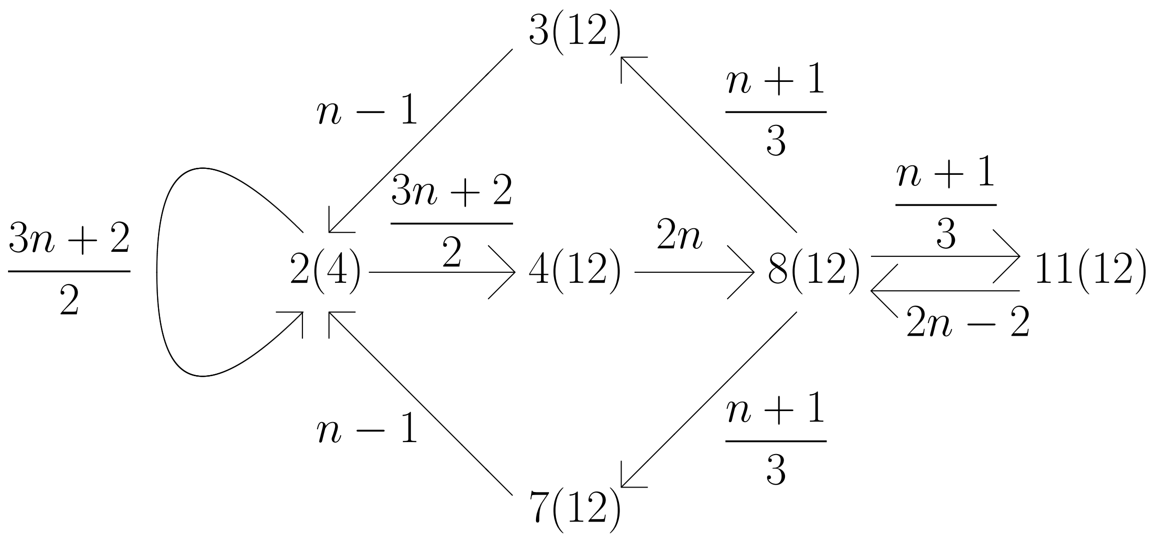
for certain $a_{r(m)}, b_{r(m)}, c_{r(m)} \in R$.

Our group elements will be **bijective** residue class-wise affine mappings, i.e. we require that $a_{r(m)} \neq 0$ and that the residue classes

$$\frac{a_{r(m)} \cdot r + b_{r(m)}}{c_{r(m)}} \left(\frac{a_{r(m)} \cdot m}{c_{r(m)}} \right),\ r(m) \in R/mR$$

form a partition of R.

A Small Example



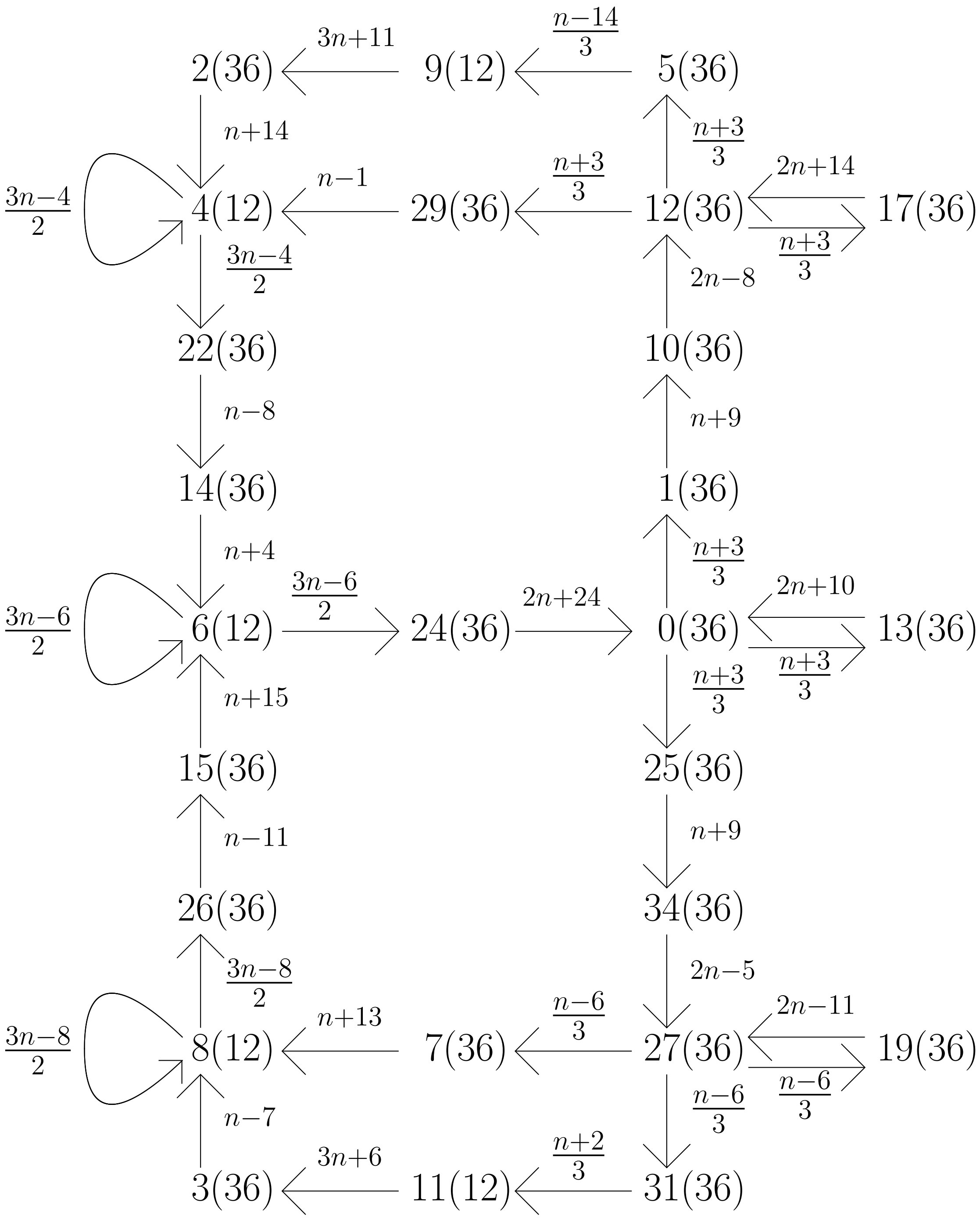
$\kappa \in \mathrm{Sym}(\mathbb{Z})$:

$$n \longmapsto \begin{cases} (3n+2)/2 & \text{if } n \in 2(4), \\ (n+1)/3 & \text{if } n \in 8(12), \\ 2n & \text{if } n \in 4(12), \\ 2n-2 & \text{if } n \in 11(12), \\ n-1 & \text{if } n \in 3(12) \cup 7(12), \\ n & \text{otherwise.} \end{cases}$$

For any $k \in \mathbb{N}$ the set of integers belonging to cycles of κ of length $l = 3k+1$ is

$$\mathcal{C}_k := \bigcup \left(\left(2(4) \setminus \bigcup_{j=1}^{k-1} \mathcal{C}_j \right) \setminus \bigcup_{j=0}^{k} \left(2(4)^{\kappa^j} \cap 2(4)^{\kappa^{-(k-j)}} \right) \right)^{\langle \kappa \rangle}.$$

A Cycle Traversing Almost All Integers



Definition: We denote the group of all residue class-wise affine permutations of the integers by $\mathrm{RCWA}(\mathbb{Z})$.

Let $S \subseteq \mathbb{Z}$ be a set-theoretic union of finitely many residue classes. The largest subgroup of $\mathrm{RCWA}(\mathbb{Z})$ whose support is S is permutation-isomorphic to $\mathrm{RCWA}(\mathbb{Z})$ itself.

Given an injective residue class-wise affine mapping f whose image is S, the corresponding *restriction monomorphism* π_f is a permutation isomorphism from $\mathrm{RCWA}(\mathbb{Z})$ to this subgroup: We set

$$\pi_f : \mathrm{RCWA}(\mathbb{Z}) \hookrightarrow \mathrm{RCWA}(\mathbb{Z}), \quad \sigma \mapsto \sigma_f,$$

where σ_f is the uniquely determined mapping which fixes the complement of S pointwise and makes the following diagram commutative:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\ \sigma\ } & \mathbb{Z} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbb{Z} & \xrightarrow[\ \sigma_f\ ]{} & \mathbb{Z} \end{array}$$

A Set of Generators(?)

Conjecture: There is a set of generators of RCWA($\mathbb{Z}$) consisting of 3 infinite series arising as images of the mappings $\nu : n \mapsto n+1$, $\varsigma : n \mapsto -n$ resp. $\tau : n \mapsto n + (-1)^n$ under suitable restriction monomorphisms:

$$\nu_{r(m)}: \ n \mapsto \begin{cases} n+m & \text{if} \ \ n \in r(m), \\ n & \text{otherwise,} \end{cases}$$

$$\varsigma_{r(m)}: \ n \mapsto \begin{cases} -n+2r & \text{if} \ \ n \in r(m), \\ n & \text{otherwise} \end{cases}$$

and

$\tau_{r_1(m_1),r_2(m_2)}:$

$$n \mapsto \begin{cases} \dfrac{m_2 n + (m_1 r_2 - m_2 r_1)}{m_1} & \text{if} \ \ n \in r_1(m_1), \\ \dfrac{m_1 n + (m_2 r_1 - m_1 r_2)}{m_2} & \text{if} \ \ n \in r_2(m_2), \\ n & \text{otherwise,} \end{cases}$$

where $r(m)$ runs over the residue classes of $\mathbb{Z}$ and $r_1(m_1)$ and $r_2(m_2)$ run over the pairs of disjoint residue classes of $\mathbb{Z}$.

Example: A factorization of κ into involutions is given by

$$\kappa = \tau_{2(4),3(4)} \cdot \tau_{3(4),8(12)} \cdot \tau_{4(6),8(12)}.$$

We have

$$\tau_{2(4),3(4)}: \ n \longmapsto \begin{cases} n+1 & \text{if } n \in 2(4), \\ n-1 & \text{if } n \in 3(4), \\ n & \text{otherwise,} \end{cases}$$

$$\tau_{3(4),8(12)}: \ n \longmapsto \begin{cases} 3n-1 & \text{if } n \in 3(4), \\ (n+1)/3 & \text{if } n \in 8(12), \\ n & \text{otherwise,} \end{cases}$$

$$\tau_{4(6),8(12)}: \ n \longmapsto \begin{cases} 2n & \text{if } n \in 4(6), \\ n/2 & \text{if } n \in 8(12), \\ n & \text{otherwise.} \end{cases}$$

Larger example: Factoring Collatz' permutation

$$\alpha \in \mathrm{RCWA}(\mathbb{Z}): \quad n \ \mapsto \ \begin{cases} 3n/2 & \text{if } \ n \in 0(2), \\ (3n+1)/4 & \text{if } \ n \in 1(4), \\ (3n-1)/4 & \text{if } \ n \in 3(4) \end{cases}$$

into the generators from the 3 series mentioned in the conjecture is **MUCH** harder – we have

$$\begin{aligned} &\tilde{\alpha} \cdot \nu^{-4} \cdot \tau_{3(144),139(288)} \cdot \tau_{75(144),235(288)} \cdot \tau_{101(144),43(288)} \\ &\quad \cdot \tau_{27(36),23(72)} \cdot \tau_{17(36),47(72)} \cdot \tau_{70(72),71(144)} \cdot \tau_{65(72),143(144)} \\ &\quad \cdot \tau_{29(144),91(288)} \cdot \tau_{27(36),70(72)} \cdot \tau_{17(36),3(72)} \cdot \tau_{29(72),187(288)} \\ &\quad \cdot \tau_{65(72),283(288)} \cdot \tau_{3(36),8(72)} \cdot \tau_{5(36),32(72)} \cdot \tau_{15(36),56(72)} \\ &\quad \cdot \tau_{3(36),91(288)} \cdot \tau_{5(36),187(288)} \cdot \tau_{15(36),283(288)} \cdot \tau_{23(24),7(48)} \\ &\quad \cdot \tau_{8(24),33(48)} \cdot \tau_{13(24),43(96)} \cdot \tau_{17(36),91(288)} \cdot \tau_{29(36),283(288)} \\ &\quad \cdot \tau_{4(12),20(24)} \cdot \tau_{21(24),19(48)} \cdot \tau_{29(36),283(288)} \cdot \tau_{3(36),1(48)} \\ &\quad \cdot \tau_{15(36),25(48)} \cdot \tau_{27(36),11(48)} \cdot \tau_{5(36),35(48)} \cdot \tau_{17(36),36(48)} \\ &\quad \cdot \tau_{29(36),9(48)} \cdot \tau_{33(48),91(288)} \cdot \tau_{20(24),187(288)} \cdot \tau_{7(48),283(288)} \\ &\quad \cdot \tau_{1(6),0(8)} \cdot \tau_{5(6),4(8)} \cdot \tau_{0(4),1(6)} \cdot \tau_{2(4),5(6)} \cdot \tau_{2(6),1(12)} \cdot \tau_{4(6),7(12)} \cdot \nu^{4} \\ &\quad \cdot \left(\tau_{1(6),0(8)} \cdot \tau_{5(6),4(8)} \cdot \tau_{0(4),1(6)} \cdot \tau_{2(4),5(6)} \cdot \tau_{2(6),1(12)} \cdot \tau_{4(6),7(12)}\right)^{4}, \end{aligned}$$

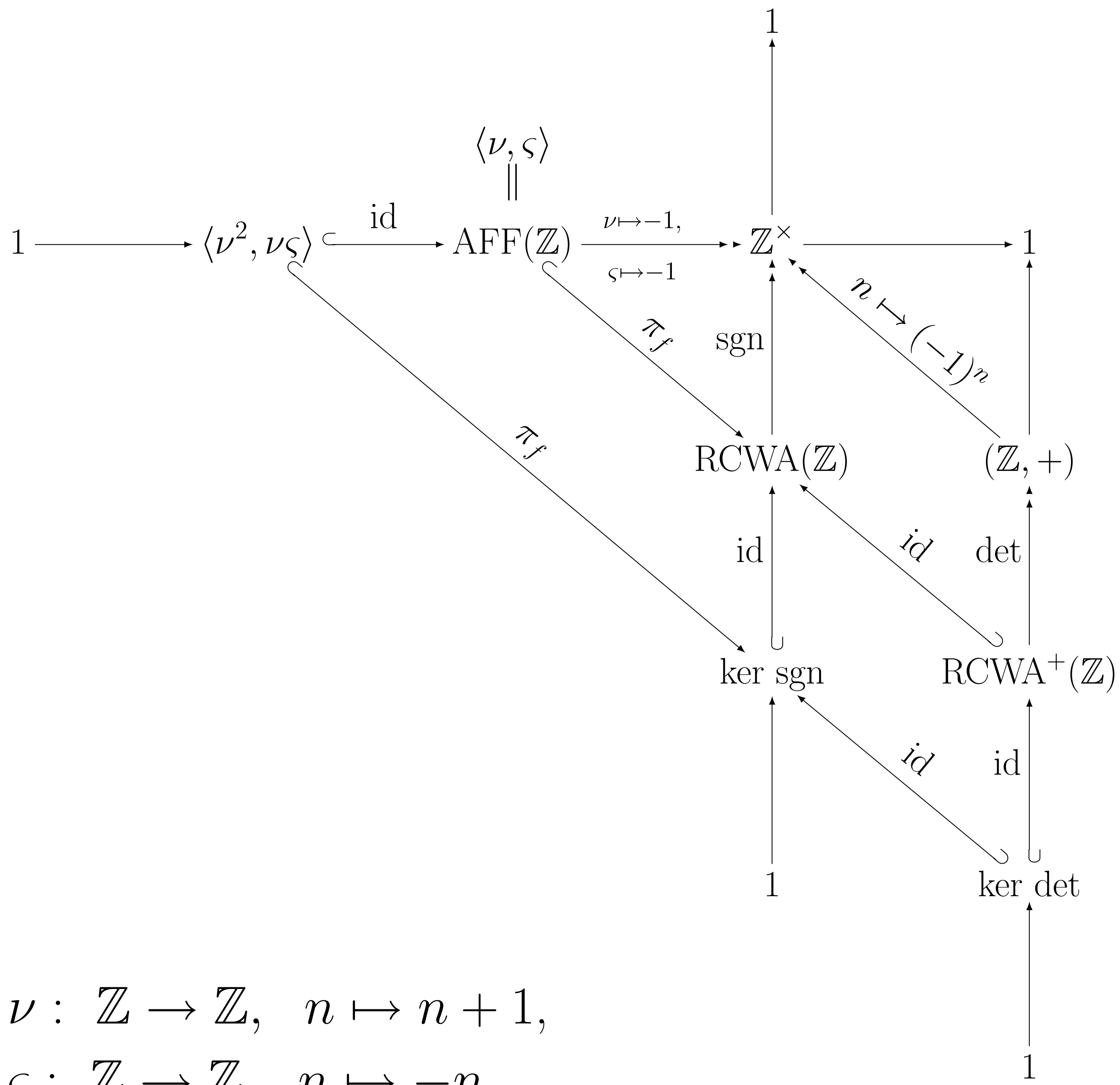
where $\tilde{\alpha}$ is affine on residue classes (mod 288), has order 101616 and can be factored into 'class transpositions' $\tau_{r_1(m_1),r_2(m_2)}$ 'just like any permutation of a finite set can be factored into transpositions'.

Remark: It is an unsolved problem whether the cycle

$$(\ldots 23, 17, 13, 10, 15, 11, 8, 12, 18, 27, 20, 30 \ldots)$$

of α is finite or infinite.

About the Structure of RCWA($\mathbb{Z}$)



$\nu: \ \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto n+1,$

$\varsigma: \ \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto -n,$

f: an injective residue class-wise affine mapping.

Vertical / horizontal sequences are short exact.

Determinant and Sign Epimorphism

If the restriction $\sigma|_{r(m)}$ of $\sigma \in \mathrm{RCWA}(\mathbb{Z})$ to some residue class $r(m) \in \mathbb{Z}/m\mathbb{Z}$ is given by $n \mapsto (a_{r(m)}n + b_{r(m)})/c_{r(m)}$, we have

$$\det(\sigma) = \frac{1}{m} \sum_{r(m)\in\mathbb{Z}/m\mathbb{Z}} \frac{b_{r(m)}}{|a_{r(m)}|}$$

and

$$\mathrm{sgn}(\sigma) = (-1)^{\det(\sigma) + \frac{1}{m} \sum_{r(m):\ a_{r(m)}<0} (m-2r)}.$$

Other characterization of determinant and sign: We set

$$\begin{aligned}
\det(\nu_{r(m)}) &:= 1, \\
\det(\tau_{r_1(m_1),r_2(m_2)}) &:= 0, \\
\mathrm{sgn}(\nu_{r(m)}) &:= \text{-}1, \\
\mathrm{sgn}(\varsigma_{r(m)}) &:= \text{-}1, \\
\mathrm{sgn}(\tau_{r_1(m_1),r_2(m_2)}) &:= 1.
\end{aligned}$$

Original motivation: Collatz Conjecture

In the 1930's, Lothar Collatz asked whether iterated application of the mapping

$$T: \mathbb{Z} \to \mathbb{Z}, \quad n \mapsto \begin{cases} n/2 & \text{if } n \text{ even}, \\ (3n+1)/2 & \text{otherwise} \end{cases}$$

to any positive integer eventually leads to 1.

For example, starting with $n = 15$ we get the sequence

$$23, 35, 53, 80, 40, 20, 10, 5, 8, 4, 2, 1.$$

This problem is still open.

It already has been attacked with methods from various different parts of mathematics.

In its November 10, 2004 version, Jeffrey C. Lagarias' commented bibliography on Collatz' conjecture lists 193 corresponding citations.

None of them uses a group theoretic approach or investigates groups generated by 'Collatz-like', hence residue class-wise affine, mappings.