# Simple Groups Generated by Involutions Interchanging Residue Classes of the Integers

Stefan Kohl

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# The Group $\mathrm{CT}(\mathbb{Z})$

By $r(m)$ we denote the residue class $r + m\mathbb{Z}$.

Let $r_1(m_1)$ and $r_2(m_2)$ be disjoint residue classes of $\mathbb{Z}$. Recall that this means that $\gcd(m_1, m_2) \nmid (r_1 - r_2)$.

We always assume that $0 \leqslant r_1 < m_1$ and that $0 \leqslant r_2 < m_2$.

Let the *class transposition* $\tau_{r_1(m_1),r_2(m_2)}$ be the permutation which interchanges $r_1 + tm_1$ and $r_2 + tm_2$ for every $t \in \mathbb{Z}$, and which fixes everything else.

For convenience, we set

$$\tau := \tau_{0(2),1(2)} : \ n \mapsto n + (-1)^n.$$

Let $\mathrm{CT}(\mathbb{Z})$ be the group which is generated by *all* class transpositions of $\mathbb{Z}$.

## Basic Properties of $\mathrm{CT}(\mathbb{Z})$

The group $\mathrm{CT}(\mathbb{Z})$ is simple.

It is countable, but it has an uncountable series of simple subgroups $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$, which is parametrized by the sets $\mathbb{P}$ of odd primes.

Further, the group $\mathrm{CT}(\mathbb{Z})$

- is not finitely generated,
- acts highly transitively on $\mathbb{N}_0$, and
- its torsion elements are divisible.

## Some Groups Which Embed into $\mathrm{CT}(\mathbb{Z})$

- Every finite group embeds into $\mathrm{CT}(\mathbb{Z})$.
- Every free group of finite rank embeds into $\mathrm{CT}(\mathbb{Z})$.
- Every free product of finitely many finite groups embeds into $\mathrm{CT}(\mathbb{Z})$.
- The class of subgroups of $\mathrm{CT}(\mathbb{Z})$ is closed under taking
  - direct products,
  - wreath products with finite groups, and
  - restricted wreath products with $(\mathbb{Z}, +)$.

## More on Subgroups of $\mathrm{CT}(\mathbb{Z})$

The group $\mathrm{CT}(\mathbb{Z})$ has

- finitely generated subgroups which do not have finite presentations, and

- finitely generated subgroups with unsolvable membership problem.

Since words in the generators of subgroups of $\mathrm{CT}(\mathbb{Z})$ can always be evaluated and compared, groups with unsolvable word problem do *not* embed into $\mathrm{CT}(\mathbb{Z})$.

## Examples of Subgroups of $\mathrm{CT}(\mathbb{Z})$

We have for example

- $\mathrm{F}_2 \cong \langle (\tau \cdot \tau_{0(2),1(4)})^2, (\tau \cdot \tau_{0(2),3(4)})^2 \rangle$ (the free group of rank 2),
- $\mathrm{PSL}(2,\mathbb{Z}) \cong \langle \tau, \tau_{0(4),2(4)} \cdot \tau_{1(2),0(4)} \rangle$ (the modular group),
- $\mathrm{C}_2 \wr \mathbb{Z} \cong \langle \tau \cdot \tau_{0(2),1(4)}, \tau_{3(8),7(8)} \rangle$ (the lamplighter group), and
- $\mathbb{Z} \wr \mathbb{Z} \cong \langle \tau \cdot \tau_{0(2),1(4)}, \tau_{3(8),7(8)} \cdot \tau_{3(8),7(16)} \rangle$, and
- $G := \langle \tau_{0(4),3(4)}, \tau_{0(6),3(6)}, \tau_{1(4),0(6)} \rangle$ is an infinite group, which has only finite orbits on $\mathbb{Z}$.

## Products of Two Class Transpositions

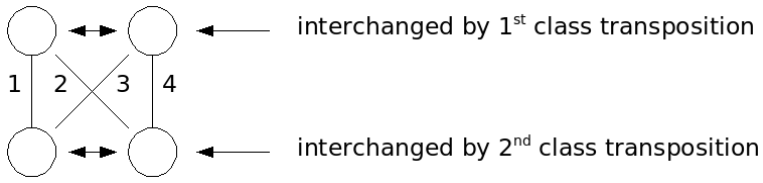
Some examples:

| $\sigma$ | ord($\sigma$) |
|---|---|
| $\tau_{0(4),2(4)} \cdot \tau_{1(4),3(4)}$ | 2 |
| $\tau_{0(3),1(3)} \cdot \tau_{0(3),2(3)}$ | 3 |
| $\tau_{0(2),1(2)} \cdot \tau_{0(4),2(4)}$ | 4 |
| $\tau_{1(2),0(4)} \cdot \tau_{1(4),2(4)}$ | 6 |
| $\tau_{0(2),1(4)} \cdot \tau_{2(3),1(6)}$ | 10 |
| $\tau_{1(2),0(4)} \cdot \tau_{1(3),2(6)}$ | 12 |
| $\tau_{0(2),1(4)} \cdot \tau_{0(3),2(3)}$ | 15 |
| $\tau_{0(3),1(6)} \cdot \tau_{1(4),3(4)}$ | 20 |
| $\tau_{0(2),1(4)} \cdot \tau_{0(5),2(5)}$ | 30 |
| $\tau_{1(3),0(6)} \cdot \tau_{1(5),2(5)}$ | 60 |
| $\tau_{0(4),1(6)} \cdot \tau_{1(4),2(6)}$ | $\infty$, finite cycles |
| $\tau_{0(2),1(4)} \cdot \tau_{1(2),2(4)}$ | $\infty$, infinite cycles |

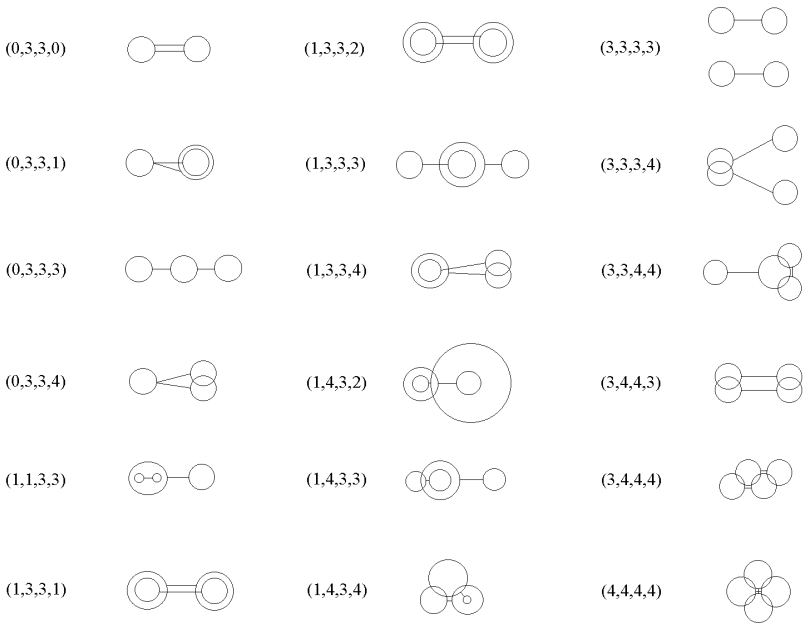
Already for class transpositions which interchange residue classes with moduli $\leqslant 6$, there are 88 different subcases where the products have different cycle structure.

# Intersection Types

On this slide, circles denote residue classes.




Residue classes interchanged by the class transpositions are connected by lines:



## On the Series of Subgroups $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$

Let $\mathbb{P}$ be a set of odd primes.

The group $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is the subgroup of $\mathrm{CT}(\mathbb{Z})$ which is generated by all class transpositions $\tau_{r_1(m_1),r_2(m_2)}$ for which all odd prime factors of $m_1$ and $m_2$ lie in $\mathbb{P}$.

The groups $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ are simple as well.

*Question:* Are the uncountably many groups $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ pairwise nonisomorphic?
If not: Under which conditions on the sets $\mathbb{P}_1$ and $\mathbb{P}_2$ of odd primes is $\mathrm{CT}_{\mathbb{P}_1}(\mathbb{Z}) \cong \mathrm{CT}_{\mathbb{P}_2}(\mathbb{Z})$?

## More on $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$

The group $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is finitely generated if and only if the set $\mathbb{P}$ is finite.

The intersection of all groups $\mathrm{CT}_{\mathbb{P}}(\mathbb{Z})$ is $\mathrm{CT}_{\emptyset}(\mathbb{Z})$. We have

$$\mathrm{CT}_{\emptyset}(\mathbb{Z}) \;=\; \langle \kappa, \lambda, \mu, \nu \rangle,$$

where $\kappa = \tau_{0(2),1(2)}$, $\lambda = \tau_{1(2),2(4)}$, $\mu = \tau_{0(2),1(4)}$ and $\nu = \tau_{1(4),2(4)}$.

John McDermott (Galway) has pointed out to me the following:

The group $\mathrm{CT}_{\emptyset}(\mathbb{Z})$ is isomorphic to the Higman-Thompson group (cf. Higman 1974), the first finitely presented infinite simple group which has been discovered.

## More on $\mathrm{CT}_{\emptyset}(\mathbb{Z})$

To check that the group $\mathrm{CT}_{\emptyset}(\mathbb{Z})$ is isomorphic to the Higman-Thompson group, it suffices to verify that its generators satisfy the relations given by Higman:

- $\kappa^2 = \lambda^2 = \mu^2 = \nu^2 = 1$,
- $\lambda\kappa\mu\kappa\lambda\nu\kappa\nu\mu\kappa\lambda\kappa\mu = \kappa\nu\lambda\kappa\mu\nu\kappa\lambda\nu\mu\nu\lambda\nu\mu = 1$,
- $(\lambda\kappa\mu\kappa\lambda\nu)^3 = (\mu\kappa\lambda\kappa\mu\nu)^3 = 1$,
- $(\lambda\nu\mu)^2\kappa(\mu\nu\lambda)^2\kappa = 1$,
- $(\lambda\nu\mu\nu)^5 = 1$,
- $(\lambda\kappa\nu\kappa\lambda\nu)^3\kappa\nu\kappa(\mu\kappa\nu\kappa\mu\nu)^3\kappa\nu\kappa\nu = 1$,
- $((\lambda\kappa\mu\nu)^2(\mu\kappa\lambda\nu)^2)^3 = 1$,
- $(\lambda\nu\lambda\kappa\mu\kappa\mu\nu\lambda\nu\mu\kappa\mu\kappa)^4 = 1$,
- $(\mu\nu\mu\kappa\lambda\kappa\lambda\nu\mu\nu\lambda\kappa\lambda\kappa)^4 = 1$, and
- $(\lambda\mu\kappa\lambda\kappa\mu\lambda\kappa\nu\kappa)^2 = (\mu\lambda\kappa\mu\kappa\lambda\mu\kappa\nu\kappa)^2 = 1$.

## Simple Supergroups of $\mathsf{CT}(\mathbb{Z})$

Let $r(m) \subseteq \mathbb{Z}$ be a residue class.

We define the *class shift* $\nu_{r(m)}$ by

$$\nu_{r(m)} \in \mathsf{Sym}(\mathbb{Z}) : \; n \mapsto \begin{cases} n+m & \text{if } \; n \in r(m), \\ n & \text{otherwise.} \end{cases}$$

We define the *class reflection* $\varsigma_{r(m)}$ by

$$\varsigma_{r(m)} \in \mathsf{Sym}(\mathbb{Z}) : \; n \mapsto \begin{cases} -n+2r & \text{if } \; n \in r(m), \\ n & \text{otherwise,} \end{cases}$$

where we assume that $0 \leqslant r < m$.

The groups

$$K^+ := \langle \mathsf{CT}(\mathbb{Z}), \nu_{1(3)} \cdot \nu_{2(3)}^{-1} \rangle$$

and

$$K^- := \langle \mathsf{CT}(\mathbb{Z}), \nu_{1(3)} \cdot \nu_{2(3)}, \varsigma_{0(2)} \cdot \nu_{0(2)} \rangle$$

are simple as well.

## Computational Aspects

So far, research in computational group theory focussed mainly on finite permutation groups, matrix groups, finitely presented groups, polycyclically presented groups and automatic groups.

The subgroups of $\mathrm{CT}(\mathbb{Z})$ form another large class of groups which are accessible to computational methods. Algorithms to compute with such groups are described in

Algorithms for a Class of Infinite Permutation Groups. *J. Symb. Comput.* 43(2008), no. 8, 545-581.

They are implemented in the package RCWA for the computer algebra system GAP .

Many of the results presented in this talk have first been discovered during extensive experiments with the RCWA package.

## A Little Example

In 1932, Lothar Collatz investigated the permutation

$$\alpha: \quad n \mapsto \begin{cases} 2n/3 & \text{if } n \in 0(3), \\ (4n-1)/3 & \text{if } n \in 1(3), \\ (4n+1)/3 & \text{if } n \in 2(3) \end{cases}$$

of the integers. The cycle structure of $\alpha$ is unknown so far.

We want to determine whether $\alpha \in \mathsf{CT}(\mathbb{Z})$.

For this, we attempt to factor $\alpha$ into class transpositions. Due to the particular form of $\alpha$, that is not particularly easy and we need a notable number of factors.

## "Prime Switch" $\sigma_p$

The factorization method makes use of certain special products of class transpositions:

For an odd prime $p$, let

$$\begin{aligned}\sigma_p := {} & \tau_{0(8),1(2p)} \cdot \tau_{4(8),2p-1(2p)} \\ & \cdot \tau_{0(4),1(2p)} \cdot \tau_{2(4),2p-1(2p)} \\ & \cdot \tau_{2(2p),1(4p)} \cdot \tau_{4(2p),2p+1(4p)} \in \mathsf{CT}(\mathbb{Z}).\end{aligned}$$

We have

$$\sigma_p : \; n \mapsto \begin{cases} (pn+2p-2)/2 & \text{if } n \in 2(4), \\ n/2 & \text{if } n \in 0(4) \setminus (4(4p) \cup 8(4p)), \\ n+2p-7 & \text{if } n \in 8(4p), \\ n-2p+5 & \text{if } n \in 2p-1(2p), \\ n+1 & \text{if } n \in 1(2p), \\ n-3 & \text{if } n \in 4(4p), \\ n & \text{if } n \in 1(2) \setminus (1(2p) \cup 2p-1(2p)). \end{cases}$$

## $\alpha \in \mathsf{CT}(\mathbb{Z})$

Now we have

$$\begin{aligned}
\alpha \;=\;& \tau_{2(3),3(6)} \cdot \tau_{1(3),0(6)} \cdot \tau_{0(3),1(3)} \cdot \tau \cdot \tau_{0(36),1(36)} \\
&\cdot \tau_{0(36),35(36)} \cdot \tau_{0(36),31(36)} \cdot \tau_{0(36),23(36)} \cdot \tau_{0(36),18(36)} \\
&\cdot \tau_{0(36),19(36)} \cdot \tau_{0(36),17(36)} \cdot \tau_{0(36),13(36)} \cdot \tau_{0(36),5(36)} \\
&\cdot \tau_{2(36),10(36)} \cdot \tau_{2(36),11(36)} \cdot \tau_{2(36),15(36)} \cdot \tau_{2(36),20(36)} \\
&\cdot \tau_{2(36),28(36)} \cdot \tau_{2(36),26(36)} \cdot \tau_{2(36),25(36)} \cdot \tau_{2(36),21(36)} \\
&\cdot \tau_{2(36),4(36)} \cdot \tau_{3(36),8(36)} \cdot \tau_{3(36),7(36)} \cdot \tau_{9(36),16(36)} \\
&\cdot \tau_{9(36),14(36)} \cdot \tau_{9(36),12(36)} \cdot \tau_{22(36),34(36)} \\
&\cdot \tau_{27(36),32(36)} \cdot \tau_{27(36),30(36)} \cdot \tau_{29(36),33(36)} \\
&\cdot \tau_{10(18),35(36)} \cdot \tau_{5(18),35(36)} \cdot \tau_{10(18),17(36)} \\
&\cdot \tau_{5(18),17(36)} \cdot \tau_{8(12),14(24)} \cdot \tau_{6(9),17(18)} \cdot \tau_{3(9),17(18)} \\
&\cdot \tau_{0(9),17(18)} \cdot \tau_{6(9),16(18)} \cdot \tau_{3(9),16(18)} \cdot \tau_{0(9),16(18)} \\
&\cdot \tau_{6(9),11(18)} \cdot \tau_{3(9),11(18)} \cdot \tau_{0(9),11(18)} \cdot \tau_{6(9),4(18)} \\
&\cdot \tau_{3(9),4(18)} \cdot \tau_{0(9),4(18)} \cdot \tau_{0(6),14(24)} \cdot \tau_{0(6),2(24)} \\
&\cdot \tau_{8(12),17(18)} \cdot \tau_{7(12),17(18)} \cdot \tau_{8(12),11(18)} \\
&\cdot \tau_{7(12),11(18)} \cdot \sigma_3^{-1} \cdot \tau_{7(12),17(18)} \cdot \tau_{2(6),17(18)} \\
&\cdot \tau_{0(3),17(18)} \cdot \sigma_3^{-3} \in \mathsf{CT}(\mathbb{Z}).
\end{aligned}$$

## The $3n+1$ Conjecture

In the 1930s, Lothar Collatz made the following conjecture:

**3n+1 Conjecture.** Iterated application of the mapping

$$T : \mathbb{Z} \to \mathbb{Z},\ n \mapsto \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (3n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

to any positive integer yields 1 after a finite number of steps.

This conjecture – nowadays famous – is still open today, although there are more than 200 related mathematical publications. - Cf. Jeffrey C. Lagarias' annotated bibliography (http://arxiv.org/abs/math.NT/0309224, http://arxiv.org/abs/math.NT/0608208).

## A Bijective Extension of $T$ to $\mathbb{Z}^2$

The mapping $T$ is not injective.

Dealing with permutations and permutation groups is usually easier.

However, the mapping $T$ can be extended in natural ways to permutations of $\mathbb{Z}^2$. –

For example:

$\sigma_T \in \mathsf{Sym}(\mathbb{Z}^2)$ :

$$(m,n) \mapsto \begin{cases} (2m+1,(3n+1)/2) & \text{if } \ n \in 1(2), \\ (2m,n/2) & \text{if } \ n \in 4(6), \\ (m,n/2) & \text{otherwise.} \end{cases}$$

This turns the $3n+1$ conjecture into the question whether the line $n = 4$ is a set of representatives for the cycles of $\sigma_T$ on the half-plane $n > 0$.

## A Factorization of $\sigma_T$

Furthermore, the mapping $\sigma_T$ can be written as the product of two permutations whose cycle structure can be described very easily:

We have $\sigma_T = \alpha\beta$, where

$$\alpha : (m,n) \mapsto \begin{cases} (2m, n/2) & \text{if } 2|n, \\ (2m+1, (n-1)/2) & \text{if } 2 \nmid n, \end{cases}$$

and

$$\beta : (m,n) \mapsto \begin{cases} (m/2, n) & \text{if } 2|m \text{ and } n \notin 2(3), \\ (m,n) & \text{if } 2|m \text{ and } n \in 2(3), \\ (m, 3n+2) & \text{if } 2 \nmid m. \end{cases}$$

This motivates a move from $\mathbb{Z}$ to $\mathbb{Z}^2$, and generalizing further, to $\mathbb{Z}^d$ for $d \in \mathbb{N}$.

## The Groups $\mathrm{CT}(\mathbb{Z}^d)$

Let $d \in \mathbb{N}$, and let $L_1, L_2 \in \mathbb{Z}^{d\times d}$ be matrices of full rank which are in Hermite normal form.

Further let $r_1+\mathbb{Z}^d L_1$ and $r_2+\mathbb{Z}^d L_2$ be disjoint residue classes, and assume that $r_1$ and $r_2$ are reduced modulo $\mathbb{Z}^d L_1$ and $\mathbb{Z}^d L_2$, respectively.

Let the *class transposition*

$$\tau_{r_1+\mathbb{Z}^d L_1, r_2+\mathbb{Z}^d L_2} \in \mathrm{Sym}(\mathbb{Z}^d)$$

be the involution which interchanges $r_1+kL_1$ and $r_2+kL_2$ for every $k \in \mathbb{Z}^d$, and which fixes everything else.

Let $\mathrm{CT}(\mathbb{Z}^d)$ be the group which is generated by the set of all class transpositions of $\mathbb{Z}^d$.

The groups $\mathrm{CT}(\mathbb{Z}^d)$, $d \in \mathbb{N}$ are simple as well.

The development version of RCWA contains already basic methods to compute in $\mathrm{CT}(\mathbb{Z}^2)$.

## Recent Paper

Many of the results presented in this talk are included in the paper

A Simple Group Generated by Involutions Interchanging Residue Classes of the Integers. *Mathematische Zeitschrift*, DOI: 10.1007/s00209-009-0497-8.

The GAP package RCWA is available at

http://www.gap-system.org/Packages/rcwa.html