Final Test in MAT 410: Introduction to Topology
Answers to the Test Questions

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Question 1: Give the definition of a *topological space*. (3 credits)

Answer: A *topological space* (X, τ) is a pair consisting of a set X and a collection τ of subsets of X such that the following hold:

- $\{\emptyset, X\} \subset \tau$.
- τ is closed under taking arbitrary unions and finite intersections.

The sets in τ are termed *open* sets.

Question 2: Let X and Y be topological spaces. Describe under which condition a function $f : X \to Y$ is said to be

1. *continuous*,
2. an *identification map*.

(3 credits – 1 for (1.) and 2 for (2.))

Answer:

1. If and only if preimages of open sets are open.
2. If and only if it is surjective and the open sets in Y are precisely the images of the open sets in X.

Question 3: Let X and Y be topological spaces. Give the definition of the *product topology* on $X \times Y$. (3 credits)

Answer: $U \subseteq X \times Y$ is open in the *product topology* if and only if, given any point $(x, y) \in U$, there are open sets V in X and W in Y such that $x \in V$, $y \in W$ and $V \times W \subseteq U$.

Question 4: Give the definition of a *quotient topology*, and – considering different kinds of quotient structures you know from other parts of mathematics – explain why "quotient" topology is a reasonably chosen mathematical term. (4 credits – 2 of them for the explanation)

Answer: Let X be a topological space, Y be a set and $f : X \to Y$ be a surjection. Then the *quotient topology* on Y is the unique topology with which f becomes an identification map.

As with quotients of groups, rings etc., in a quotient space, points / elements belonging to the same equivalence class with respect to a certain equivalence relation are identified with each other. – So the use of the term 'quotient' for topological spaces is in line with its use for different mathematical structures.

Question 5: State when a topological space is said to be

1. *compact*,
2. *connected*.

(4 credits)

Answer: A topological space is said to be

1. *compact* if every open cover possesses a finite subcover, and
2. *connected* if it admits no nontrivial partition into open sets.

Question 6: Give the definition of the *diameter* of a subset of a metric space. (2 credits)

Answer: The *diameter* of a subset A of a metric space (X, d) is $\sup\{d(x, y) \mid (x, y) \in A \times A\}$.

Question 7: What is the difference between the Klein bottle and the torus? – Explain. (3 credits)

Answer: The torus is an oriented surface, and it can be embedded without self-intersection into $\mathbb{R}^3$. The Klein bottle is a non-oriented surface which cannot be embedded without self-intersection into $\mathbb{R}^3$.

Question 8: Let $\mathbb{R}^2$ be endowed with the usual topology. Either prove or disprove that $[0, 1[\times]0, 1[$ and $[0, 1[\times[0, 1]$ are homeomorphic subspaces of $\mathbb{R}^2$. (8 credits)

Answer: A nice proof (with illustrations, which I don't want to copy here) that they are homeomorphic can be found at

http://www-history.mcs.st-and.ac.uk/ john/MT4522/Solutions/S6.7.html

For getting the credits, you were supposed to show that you have understood in principle in what way a homeomorphism can map $[0, 1[\times]0, 1[$ to $[0, 1[\times[0, 1]$.

Explicitly determining an homeomorphism would be too time-consuming for an in-class examination, and was therefore not required for getting the full number of credits. One possibility to construct one is to partition source and range in a suitable way into 6 triangles, each, and to use linear algebra to find 6 affine mappings which map the triangles of the source to the corresponding triangles of the range. This requires setting up and solving 6 systems of 6 linear equations, each. Of course there are many other ways to construct such homeomorphisms.

Question 9: Let $\mathbb{R}^3$ be endowed with the usual topology, and let

1. $A := \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 0\}$,
2. $B := \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 1\}$,
3. $C := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 0\}$,
4. $D := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ and
5. $E := \{(x, y, z) \in \mathbb{R}^3 \mid |x| + |y| + |z| \in \mathbb{Q}\}$

be endowed with the respective subspace topologies. Find out which of the topological spaces A, B, C, D and E are homeomorphic (if any), and which are not. – Proofs required (no credits without arguments). (10 credits)

Answer:

1. A is the union of the planes $x = 0$, $y = 0$ and $z = 0$, so it is infinite, connected and unbounded (i.e. not compact).
2. B is infinite, unbounded (i.e. not compact) and has 4 connected components (to get xyz positive, either all three variables need to be positive or exactly two of them need to be negative – and none of them may be 0).
3. $C = \{(0, 0, 0)\}$ is finite.
4. D is the unit sphere about the origin, and hence infinite, connected, closed and bounded (i.e. compact).
5. E has infinitely many connected components – one for each positive rational number.

Since cardinality, compactness and number of connected components are invariant under homeomorphism, all five sets are pairwise non-homeomorphic.

Question 10: Find out which of the following 20 assertions are true and which are false (only true/false answers – correct answer: 1 credit, no answer: 0 credits, wrong or unclear answer: -1 credit, $\geqslant 0$ credits in total; answers must be marked by an 'X' in the box after either 'true' or 'false'):

1. For every $n \in \mathbb{N}$ there is a topological space with n points.
   true (X) false ( )
   Just take $\{1, \ldots, n\}$ with the trivial topology.
2. Given $n \in \mathbb{N}$, up to homeomorphism there are exactly $5 \cdot (2^n + 2^{n-1}) - 1$ topological spaces with n points.
   true ( ) false (X)
   Counterexample: There is only one possible topology on a 1-element set – not 14.
3. Every metric space can also be seen as a topological space.
   true (X) false ( )
   Topological spaces are a generalization of metric spaces – see script.
4. Given any topological space X, one obtains another topological space $\mathcal{C}(X)$ with the same points as X – the so-called *complement space* of X – by letting the open sets in $\mathcal{C}(X)$ be the sets which are closed in X, and the closed sets in $\mathcal{C}(X)$ be the sets which are open in X.
   true ( ) false (X)
   For example in $\mathbb{R}$ with the usual topology, one-point sets are closed. So since arbitrary unions of open sets are open, in $\mathcal{C}(\mathbb{R})$ every set must be open – but in $\mathbb{R}$ not every set is closed. – Contradiction.
5. There are topological spaces with countably many points, which have uncountably many open sets.
   true (X) false ( )
   Example: countable set with the discrete topology.
6. The number of points of a finite Hausdorff space is always a prime power.
   true ( ) false (X)
   Counterexample: 6-element set with the discrete topology. – This is a Hausdorff space whose number of points is not a prime power.

7. $\mathbb{R}$ with the usual topology is a compact topological space.
   true (   ) false (X)
   The open cover $\mathbb{R} = \cup_{n\in\mathbb{Z}}]n, n+2[$ does not possess a finite subcover.

8. $\mathbb{R}$ with the Zariski topology is a compact topological space.
   true (X) false (   )
   See Exercise 29.

9. $\mathbb{R}$ with the usual topology is a connected topological space.
   true (X) false (   )
   The only sets in $\mathbb{R}$ which are both open and closed are $\emptyset$ and $\mathbb{R}$.

10. $\mathbb{R}$ with the Zariski topology is a connected topological space.
    true (X) false (   )
    No subset of $\mathbb{R}$ is both finite and has a finite complement – so the above holds also here.

11. All Hausdorff spaces with countably many points are compact.
    true (   ) false (X)
    Counterexample: $\mathbb{Z}$ with the discrete topology. – Then the open cover consisting of the 1-element subsets possesses no finite subcover.

12. In a compact metric space, every sequence of points has a convergent subsequence.
    true (X) false (   )
    See Theorem 7.22 in the script.

13. Finite topological spaces are always connected.
    true (   ) false (X)
    Counterexample: discrete topological space with at least 2 points.

14. Finite topological spaces are never connected.
    true (   ) false (X)
    Counterexample: any finite set with the trivial topology.

15. There are Hausdorff spaces which are totally disconnected.
    true (X) false (   )
    Example: discrete topological spaces.

16. Let $\mathbb{Z}$ be endowed with the topology where the open sets are the set-theoretic unions of residue classes. Then $f : \mathbb{Z} \to \mathbb{Z},\ n \mapsto n + (-1)^n$ is an homeomorphism.
    true (X) false (   )
    The function f maps unions of residue classes to unions of residue classes, so it is an homeomorphism.

17. The set $S \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ of zeros of a polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ is always bounded, and its diameter is bounded above by the product of the coefficients of P.
    true (   ) false (X)
    Counterexample: the set of zeros of the zero polynomial is $\mathbb{R}^n$, and hence unbounded.

18. Let A be a bounded subset of $\mathbb{R}^3$, and let $B \subset \mathbb{R}^3$ be a superset of A such that $B \setminus A$ is finite. Then the diameter of B is the same as the diameter of A.
    true (   ) false (X)
    B may contain a point 'far away' from A.

19. The diameter of the closure of a subset $A \subset \mathbb{R}$ is always the same as the diameter of A itself.
true (X) false (   )
See Lemma 7.19 in the script.

20. The diameter of the interior of a subset $A \subset \mathbb{R}$ is always the same as the diameter of A itself.
true (   ) false (X)
Counterexample: $A = \mathbb{Q} \cap [0, 1]$.

(20 credits)