Final Test in MAT 410: Introduction to Topology

Stefan Kohl

Date and time: Wednesday, February 10, 2010, 17:00 - 19:00.
Name:

You have **120 minutes** of time to answer the 10 questions below. Write your answers to Questions 1 – 9 to the blank sheets of paper supplied to you, and answer Question 10 on this sheet. You are not allowed to use anything else than a pen. The rules announced by e-mail apply.

Question 1: Give the definition of a *topological space*. (3 credits)

Question 2: Let X and Y be topological spaces. Describe under which condition a function $f : X \to Y$ is said to be

1. *continuous*,
2. an *identification map*.

(3 credits – 1 for (1.) and 2 for (2.))

Question 3: Let X and Y be topological spaces. Give the definition of the *product topology* on $X \times Y$. (3 credits)

Question 4: Give the definition of a *quotient topology*, and – considering different kinds of quotient structures you know from other parts of mathematics – explain why "quotient" topology is a reasonably chosen mathematical term. (4 credits – 2 of them for the explanation)

Question 5: State when a topological space is said to be

1. *compact*,
2. *connected*.

(4 credits)

Question 6: Give the definition of the *diameter* of a subset of a metric space. (2 credits)

Question 7: What is the difference between the Klein bottle and the torus? – Explain. (3 credits)

Question 8: Let $\mathbb{R}^2$ be endowed with the usual topology. Either prove or disprove that $[0,1[\times]0,1[$ and $[0,1[\times[0,1]$ are homeomorphic subspaces of $\mathbb{R}^2$. (8 credits)

Question 9: Let $\mathbb{R}^3$ be endowed with the usual topology, and let

1. $A := \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 0\}$,
2. $B := \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 1\}$,
3. $C := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 0\}$,
4. $D := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ and
5. $E := \{(x, y, z) \in \mathbb{R}^3 \mid |x| + |y| + |z| \in \mathbb{Q}\}$

be endowed with the respective subspace topologies. Find out which of the topological spaces A, B, C, D and E are homeomorphic (if any), and which are not. – Proofs required (no credits without arguments). (10 credits)

Question 10: Find out which of the following 20 assertions are true and which are false (only true/false answers – correct answer: 1 credit, no answer: 0 credits, wrong or unclear answer: -1 credit, $\geqslant 0$ credits in total; answers must be marked by an 'X' in the box after either 'true' or 'false'):

1. For every $n \in \mathbb{N}$ there is a topological space with n points.
   true (   ) false (   )

2. Given $n \in \mathbb{N}$, up to homeomorphism there are exactly $5 \cdot (2^n + 2^{n-1}) - 1$ topological spaces with n points.
true ( ) false ( )

3. Every metric space can also be seen as a topological space.
true ( ) false ( )

4. Given any topological space X, one obtains another topological space $\mathcal{C}(X)$ with the same points as X – the so-called *complement space* of X – by letting the open sets in $\mathcal{C}(X)$ be the sets which are closed in X, and the closed sets in $\mathcal{C}(X)$ be the sets which are open in X.
true ( ) false ( )

5. There are topological spaces with countably many points, which have uncountably many open sets.
true ( ) false ( )

6. The number of points of a finite Hausdorff space is always a prime power.
true ( ) false ( )

7. $\mathbb{R}$ with the usual topology is a compact topological space.
true ( ) false ( )

8. $\mathbb{R}$ with the Zariski topology is a compact topological space.
true ( ) false ( )

9. $\mathbb{R}$ with the usual topology is a connected topological space.
true ( ) false ( )

10. $\mathbb{R}$ with the Zariski topology is a connected topological space.
true ( ) false ( )

11. All Hausdorff spaces with countably many points are compact.
true ( ) false ( )

12. In a compact metric space, every sequence of points has a convergent subsequence.
true ( ) false ( )

13. Finite topological spaces are always connected.
true ( ) false ( )

14. Finite topological spaces are never connected.
true ( ) false ( )

15. There are Hausdorff spaces which are totally disconnected.
true ( ) false ( )

16. Let $\mathbb{Z}$ be endowed with the topology where the open sets are the set-theoretic unions of residue classes. Then $f : \mathbb{Z} \to \mathbb{Z},\ n \mapsto n + (-1)^n$ is an homeomorphism.
true ( ) false ( )

17. The set $S \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) of zeros of a polynomial $P \in \mathbb{R}[x_1, \dots, x_n]$ is always bounded, and its diameter is bounded above by the product of the coefficients of P.
true ( ) false ( )

18. Let A be a bounded subset of $\mathbb{R}^3$, and let $B \subset \mathbb{R}^3$ be a superset of A such that $B \setminus A$ is finite. Then the diameter of B is the same as the diameter of A.
true ( ) false ( )

19. The diameter of the closure of a subset $A \subset \mathbb{R}$ is always the same as the diameter of A itself.
true ( ) false ( )

20. The diameter of the interior of a subset $A \subset \mathbb{R}$ is always the same as the diameter of A itself.
true ( ) false ( )

(20 credits)

– Good luck!

Maximum possible number of credits: 60.
Grade = (number of credits)/6, rounded to the nearest integer.