Midterm Test in MAT 410: Introduction to Topology
Answers to the Test Questions (Variation 1)

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Question 1: Give the definition of a *topological space*. (3 credits)

Answer: A *topological space* (X, τ) is a pair consisting of a set X and a collection τ of subsets of X such that the following hold:

- $\{\emptyset, X\} \subset \tau$.
- τ is closed under taking arbitrary unions and finite intersections.

The sets in τ are termed *open* sets.

Question 2: Give the definition of a *metric space*. (3 credits)

Answer: A *metric space* (X, d) is a pair consisting of a set X and a distance function $d : X \times X \to \mathbb{R}_0^+$ such that the following hold:

- $d(x, y) = 0 \Leftrightarrow x = y$.
- $\forall x, y \in X \ d(x, y) = d(y, x)$ (Symmetry).
- $\forall x, y, z \in X \ d(x, y) + d(y, z) \geqslant d(x, z)$ (Triangle Inequality).

Question 3: Let $X := \{1, 2, 3, 4\}$ be a set of cardinality 4, and define a topology on X by letting the open sets be $\{4\}$, $\{1, 2\}$, $\{1, 2, 3\}$ and $\{2, 3, 4\}$. Either prove that X with this collection of open sets is a topological space, or list at least 4 violations of the axioms of a topological space. (4 credits)

Answer: 1. $\emptyset$ is not open, 2. X is not open, 3. the union of the open sets $\{1, 2\}$ and $\{4\}$ is not open, and 4. the intersection of the open sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$ is not open.

Question 4: Let $X := \{1, 2, 3, 4\}$ be a set of cardinality 4, and let $d : X \times X \to \mathbb{R}_0^+$ be the mapping defined by $d(1, 1) = 0$, $d(1, 2) = 1$, $d(1, 3) = 6$, $d(1, 4) = 2$, $d(2, 1) = 2$, $d(2, 2) = 0$, $d(2, 3) = 4$, $d(2, 4) = 1$, $d(3, 1) = 6$, $d(3, 2) = 4$, $d(3, 3) = 2$, $d(3, 4) = 2$, $d(4, 1) = 2$, $d(4, 2) = 1$, $d(4, 3) = 2$ and $d(4, 4) = 0$. Either prove that (X, d) is a metric space, or list at least 4 violations of the axioms of a metric space.
(4 credits)

Answer: 1. $d(3, 3) = 1$ violates the axiom that a point has distance 0 from itself, 2. $d(1, 2) = 1$ and $d(2, 1) = 2$ violate the axiom of symmetry, i.e. that the distance from a to b is always the same as the distance from b to a, 3. $d(1, 4) = 2$, $d(4, 3) = 2$ and $d(1, 3) = 6$ violate the triangle inequality, and 4. $d(2, 4) = 1$, $d(4, 3) = 2$ and $d(2, 3) = 4$ violate the triangle inequality as well.

Question 5: When is a topological space said to be a *Hausdorff space*? (2 credits)

Answer: Iff distinct points have disjoint neighbourhoods.

Question 6: Let $X := \{1, 2, 3, 4\}$. Give an example of a topology with which X becomes a Hausdorff space. (2 credits)

Answer: The discrete topology.

Question 7: Let $X_1 := \{1, 2, 3\}$ and $X_2 := \{4, 5, 6\}$ be topological spaces, where the open sets in X_1 are $\emptyset$, $\{1, 2\}$, $\{3\}$ and X_1, and the open sets in X_2 are $\emptyset$, $\{4, 6\}$, $\{5\}$ and X_2. Give an example of a homeomorphism from X_1 to X_2. (2 credits)

Answer: $\varphi : X_1 \to X_2$, $1 \mapsto 4$, $2 \mapsto 6$, $3 \mapsto 5$.

Question 8: When is a metric space X said to be *complete*? (2 credits)

Answer: Iff every Cauchy sequence of points of X converges to a point of X.

Question 9: Is a topological space always also a metric space? (2 credits)

Answer: No.

Question 10: Give an example of a topological space in which the union of infinitely many closed sets is not always closed. (2 credits)

Answer: $\mathbb{R}$ with the usual topology.

Question 11: The *Cantor set* is the subset of the unit interval $[0, 1]$ containing precisely those numbers which have a ternary numeral involving only digits 0 and 2. (Ternary digits are digits with respect to base 3, just like decimal digits are digits with respect to base 10). Determine the closure and the interior of the Cantor set in the usual topology on $\mathbb{R}$. (2 credits)

Answer: The Cantor set is closed, since it is the complement of a union of open intervals. So the closure is the Cantor set itself. The Cantor set is nowhere dense, since between any two of its points there are intervals containing only real numbers having also 1's as ternary digits. So the interior of the Cantor set is $\emptyset$.

Question 12: Which problem occurs when trying to embed the Klein bottle into $\mathbb{R}^3$? (2 credits)

Answer: The problem that it is not possible to embed the Klein bottle into $\mathbb{R}^3$ without allowing self-intersection.