Introduction to Topology, Exercise Sheet 3

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Exercise 12: Let X be a discrete metric space.

1. When exactly is a function $f : X \to X$ continuous?
2. When exactly is a function $f : X \to X$ a homeomorphism?

(2 credits)

Exercise 13: Let $d_1, d_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ denote the Euclidean distance function and the *Manhattan distance function*, respectively. By definition, d_2 maps a pair of points $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ to $\sum_{i=1}^n |x_i - y_i|$. Prove that a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous as a function from $(\mathbb{R}^n, d_1)$ to $(\mathbb{R}^n, d_1)$ if and only if it is continuous as a function from $(\mathbb{R}^n, d_2)$ to $(\mathbb{R}^n, d_2)$. (3 credits)

Exercise 14: Given $l \in \mathbb{N}$, define the distance function $d_l : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ by

$$d_l(x, y) \ := \ \sqrt[l]{\sum_{i=1}^n |x_i - y_i|^l}.$$

Let l_1 and l_2 be any positive integers. Prove that a sequence of points is a Cauchy sequence in $(\mathbb{R}^n, d_{l_1})$ if and only if it is a Cauchy sequence in $(\mathbb{R}^n, d_{l_2})$. (3 credits)

Exercise 15: Give a distance function $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}_0^+$ such that $(\mathbb{Q}, d)$ is a complete metric space. (2 credits)