MAT 452: Introduction to Algebra II
Spring 2011, Final Exam

Stefan Kohl

Date and time: Monday, June 27, 2011, 16:00 - 18:00

Question 1: Compute the following:

1. The product $(1,2,3) \cdot (2,3,4)$.
2. The conjugate $(1,2,3,4)^{(1,2,3,4,5)}$.
3. The order of the permutation $(1,2,3)(4,5,6,7)$.
4. The sign of the permutation $(1,2)(3,4,5)$.
5. The order of the group $G := \langle (1,2,3,4), (1,3) \rangle < \mathrm{S}_4$.
6. The index of the subgroup $G := \langle (1,2)(3,4) \rangle < \mathrm{S}_4$ in S_4.

(6 credits – 1 each for (1.)-(6.))

Question 2: Either prove or disprove the following assertions:

1. The product of two units of a ring is always a unit as well.
2. The sum of two units of a ring is always a unit as well.
3. The set of units of a ring forms a subring.
4. The set of units of a ring forms an ideal.
5. There is no ring in which the units form a subring or an ideal.
6. Every element of a ring can be written as a sum of 4 units.

(12 credits – 2 each for (1.)-(6.))

Question 3: Compute the following:

1. The order of the group of units of the matrix ring $\mathbb{Z}^{2\times 2}$.
2. The order of the group of units of the field $\mathbb{F}_{5^3}$.
3. The number of subfields of $\mathbb{F}_{2^{24}}$.
4. The minimal polynomial of $\sqrt{3}+\sqrt{5}$ over $\mathbb{Q}$.
5. The nontrivial Galois automorphism of the extension field $\mathbb{Q}[\sqrt{3}]$ of $\mathbb{Q}$.
6. The isomorphism type of the Galois group of the polynomial $x^3 - 7x - 7$.

(10 credits – 1 each for (1.)-(2.), and 2 each for (3.)-(6.))

Question 4: Let

$$R := \left\{ \begin{pmatrix} a+b & 4b \\ -b & a-b \end{pmatrix} \mid a,b \in \mathbb{Q} \right\} \subset \mathbb{Q}^{2\times 2}.$$

Do the following:

1. Prove that R is a subring of $\mathbb{Q}^{2\times 2}$.
2. Either prove or disprove that R is a field.

(12 credits)