MAT 551: Algebra I
Spring 2011, Final Exam, Answers

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Answer to Question 1:

1. $(7,8) \cdot (6,7) \cdot (5,6) \cdot (4,5) \cdot (3,4) \cdot (2,3) \cdot (1,2) = (1,2,3,4,5,6,7,8)$.
2. $\mathrm{sgn}((1,2)(3,4,5)(6,7,8,9)) = 1$.
3. $G = \langle (1,2)(3,4)(5,6), (1,3,5)(2,4,6) \rangle \cong \mathrm{C}_2 \times \mathrm{C}_3 \cong \mathrm{C}_6$, since the generators commute. Thus $|G| = 6$.
4. $|\,\mathrm{GL}(2,4)| = (4^2 - 1) \cdot (4^2 - 4) = 15 \cdot 12 = 180$.
5. $|\,\mathrm{PSL}(2,7)| = ((7^2 - 1) \cdot (7^2 - 7))/(6 \cdot \gcd(6,2)) = (48 \cdot 42)/(6 \cdot 2) = 168$.
6. $\mathrm{Z}(\mathrm{SL}(3,\mathbb{Z})) = 1$, since the equation $x^3 = 1$ has only one solution in $\mathbb{Z}$.

Answer to Question 2:

1. $|\,\mathrm{A}_4 \times \mathrm{D}_4| = |\,\mathrm{A}_4\,| \cdot |\mathrm{D}_4| = 12 \cdot 8 = 96$.
2. The smallest n such that $\mathrm{A}_4 \times \mathrm{D}_4$ embeds into S_n is 8, since each of the factors embeds into S_4, and $|\,\mathrm{A}_4 \times \mathrm{D}_4| \nmid |\,\mathrm{S}_7\,| = 7! = 5040$.
3. $[\mathrm{S}_8 : \mathrm{A}_4 \times \mathrm{D}_4] = |\,\mathrm{S}_8\,|/|\,\mathrm{A}_4 \times \mathrm{D}_4| = 8!/96 = 40320/96 = 420$.
4. A Sylow 2-subgroup of $\mathrm{A}_4 \times \mathrm{D}_4$ is $\mathrm{V}_4 \times \mathrm{D}_4$.
5. All semidirect products have the same order, i.e. 96, since the set of elements of any semidirect product is the cartesian product of the sets of elements of the factors.

Answer to Question 3: $G = \langle (1,2,3,4,5,6,7), (2,3,5)(4,7,6) \rangle \cong \mathrm{C}_7 \rtimes \mathrm{C}_3$.

Answer to Question 4: Every element of G either changes or fixes parity of all points, thus $\{1,3,5,7,9,11\}$ and $\{2,4,6,8,10,12\}$ are blocks for the action of G on S. Therefore the action of G on S is not primitive.

Answer to Question 5: There are two cases:

1. $a \in H$. Then $\{a,b,c,d\} \subset H$ since $b = a \cdot ab$, $c = a \cdot ac$ and $d = a \cdot ad$, thus $H = G$ and $[G : H] = 1$. A coset representative in this case is 1.
2. $a \notin H$. Then $\{a,b,c,d\} \cap H = \emptyset$, and all generators a,b,c,d of G lie in the same coset of H by the same reasoning as above. Thus $[G : H] = 2$, and coset representatives are 1 and a, i.e. $G = H \cup aH$.

Remark: in the second case there is a kind of sign homomorphism from G to C_2 whose kernel is H, where the elements of H are all products of even numbers of generators a,b,c,d of G. Thus we are in a similar situation as with transpositions generating S_n, where A_n is the subgroup consisting of products of even numbers of generators.