MAT 551: Algebra I
Spring 2011, Midterm 1, Answers

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Answer to Question 1: In $(\mathbb{Z}, +)$, $1 + 1 = 2$, and 3^2 is either $3 + 3 = 6$ (exponentiation) or $-2+3+2 = 3$ (conjugation). In S_6, $(1,2) \cdot (1,3) = (1,2,3)$, $(1,2,3,4,5)^{(2,3)(4,5)} = (1,3,2,5,4)$ and $((1,2,3,4)(5,6))^{12} = 1$.

Answer to Question 2: All except for the second are isomorphic to the given group. The first is the elementary divisors form and the third is the abelian invariants form.

Answer to Question 3:

1. The provided information is not enough to compute the group order – for example G may be the group $\mathrm{C}_2 \times \mathrm{C}_2 \times \mathrm{C}_2 \times \mathrm{C}_2$ of order 16, or G may be the infinite group $\mathrm{D}_\infty \times \mathrm{D}_\infty$, or whatsoever.
2. There is no reason why G cannot be simple. – In fact it can be simple. To take an arbitrary example, if $a = (1,2)(3,4)$, $b = (1,3)(4,5)$, $c = (1,2)(3,5)$ and $d = (1,4)(2,5)$, then $G = \mathrm{A}_5$.
3. The group G can have at most $1 + 4 + 3 \cdot 4 + 3^2 \cdot 4 = 53$ elements which can be written as products of 3 or less of the generators a, b, c, d.

Answer to Question 4: The conditions imply that for every ordered 4-tuple (p_1, p_2, p_3, p_4) of distinct points from $\{1, 2, \ldots, 11\}$ there is precisely one element of G which maps $(1,2,3,4)$ to (p_1, p_2, p_3, p_4). So the order of G equals the number of such 4-tuples, which is $11 \cdot 10 \cdot 9 \cdot 8 = 7920$.

Finally, the group G is the Mathieu group M_{11}, the smallest of the 26 sporadic simple groups.