MAT 551: Algebra I
Spring 2011, Midterm 2, Answers

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Answer to Question 1: We have

1. $(1,2,3)(4,5) \cdot (1,2,3,4,5) = (1,3,2,4)$.
2. $((1,2,3,4)(5,6,7))^5 = (1,2,3,4)(5,7,6)$.
3. $(1,2,3,4,5)^{(3,4,5)} = (1,2,4,5,3)$.
4. $\mathrm{ord}((1,2,3)(4,5)(6,7,8,9,10)) = \mathrm{lcm}(2,3,5) = 30$.
5. $\mathrm{sgn}((1,2,3)(4,5,6)) = 1$.
6. $\mathrm{sgn}((1,2,3,4)(5,6)(7,8)) = -1$.

Answer to Question 2: We have

1. $|\,\mathrm{S}_4\,| = 4! = 24$,
2. $|\,\mathrm{A}_5\,| = \frac{5!}{2} = 60$, and
3. $[\mathrm{S}_4 : \mathrm{V}_4] = \frac{24}{4} = 6$.
4. The number of conjugacy classes of S_5 is 7. Representatives are $()$, $(1,2)$, $(1,2)(3,4)$, $(1,2,3)$, $(1,2,3)(4,5)$, $(1,2,3,4)$ and $(1,2,3,4,5)$.
5. The number of conjugacy classes of elements of order 2 in S_6 is 3. Representatives are $(1,2)$, $(1,2)(3,4)$ and $(1,2)(3,4)(5,6)$.
6. The number of Sylow 2-subgroups of S_4 is 3, since except for 1 this is the only odd divisor of $|\,\mathrm{S}_4\,| = 24$. – There cannot be only one Sylow 2-subgroup since S_4 has 9 elements of order 2.

Answer to Question 3: Since G acts 2-transitively on $\{1,\dots,n\}$, for any two distinct points $a,b \in \{1,\dots,n\}$ there is some $g \in G$ such that $1^g = a$ and $2^g = b$. This implies $(1,2)^g = (a,b) \in G$. Thus G contains all transpositions. Since the transpositions generate S_n, we have $G = \mathrm{S}_n$.

Answer to Question 4: The only nontrivial divisor $d \equiv 1 \mod 5$ of $|G| = 120$ is 6, so G must have 6 Sylow 5-subgroups. The group G acts transitively on these 6 groups via conjugation, and since G is simple this action is faithful. So G embeds into S_6. We further conclude that G must embed into A_6 – if it would not, its intersection with A_6 would be a nontrivial normal subgroup. It follows $[\mathrm{A}_6 : G] = 3$. The action of A_6 on the cosets of G is transitive, and since A_6 is simple, its kernel is 1. This yields a contradiction since A_6 has order 360 and therefore does not embed into S_3.