Final Test in MAT 641: Computational Algebra I
Answers to the Test Questions

Stefan Kohl

February 9, 2010

Question 1: Give the definition of an *ideal* of a ring. (2 credits)

Answer: An *ideal* of a ring R is a subgroup of $(R, +)$ which is closed under multiplication by elements of R from the left and from the right.

Question 2: State

1. the *Ideal Membership Problem* for ideals of multivariate polynomial rings and
2. the *Implizitization Problem* for affine varieties.

(3 credits – 1 for a. and 2 for b.)

Answer:

1. The *Ideal Membership Problem* is the problem to decide whether a given polynomial lies in the ideal generated by certain given polynomials.
2. The *Implizitization Problem* for affine varieties is the problem to find an implicit representation for an affine variety for which a parametric representation is known.

Question 3: Give the definition of a *Groebner basis* of an ideal of a multivariate polynomial ring. (3 credits)

Answer: A basis of such an ideal is called a *Groebner basis* if and only if the ideal generated by the leading terms of the basis elements equals the ideal generated by the leading terms of all elements of the ideal.

Question 4: State the *Hilbert Basis Theorem.* (2 credits)

Answer: Ideals of polynomial rings in finitely many variables over a field are finitely generated.

Question 5: State the *Ascending Chain Condition* (ACC) and the *Descending Chain Condition* (DCC) for ideals of a ring. (2 credits)

Answer:

1. ACC: Every ascending chain of ideals eventually becomes stable.
2. DCC: Every descending chain of ideals eventually becomes stable.

Question 6: Give a polynomial $P \in \mathbb{R}[x, y, z]$ such that the affine variety $V(P) \subset \mathbb{R}^3$ is the unit sphere about the origin. (2 credits)

Answer: Take $P := x^2 + y^2 + z^2 - 1$.

Question 7: Let $P := 4x^3yz^2 + xyz - 7y^6z + 8z^8 + 2xy - 1 \in \mathbb{R}[x, y, z]$. Write down the polynomial P

1. in lex order,
2. in grlex order and
3. in grevlex order.

Always assume $x > y > z$. In all cases, determine the multidegree, the leading coefficient, the leading monomial and the leading term of P with respect to the respective monomial ordering. (6 credits)

Answer:

1. $P = 4x^3yz^2 + xyz + 2xy - 7y^6z + 8z^8 - 1$, multidegree: (3,1,2), leading coefficient: 4, leading monomial: x^3yz^2, leading term: $4x^3yz^2$.
2. $P = 8z^8 - 7y^6z + 4x^3yz^2 + xyz + 2xy - 1$. multidegree: (0,0,8), leading coefficient: 8, leading monomial: z^8, leading term: $8z^8$.
3. The same as (2.).

Question 8: Let $P := 4x^3y - 3xy^2z$, $a := x^2 - y^2z$, $b := xy + z^3$ and $c := y^2 - yz$.

1. Divide P by a, b and c (in this order), using lex order.
2. Find out whether or not $\{a, b, c\}$ is a Groebner basis for the ideal $\langle a, b, c\rangle$ (proof required). *Hint:* Try to divide by a, b and c in different order.

(10 credits)

Answer: The division of P by a, b and c in this order yields $P = 4xy \cdot a + (4y^2z - 3yz) \cdot b - 4z^4 \cdot c - 4yz^5 + 3yz^4$. Interchanging b and c changes the remainder to $-4z^6 + 3z^5$. Therefore the remainder does depend on the ordering of a, b and c, and so $\{a, b, c\}$ is not a Groebner basis for the ideal $\langle a, b, c\rangle$.

Question 9: Let $X := V((x^2 + y^2 + z^2 + 3)^2 - 16(x^2 + y^2)) \subset \mathbb{R}^3$.

1. Determine the points with integer coordinates on the affine variety X. – How many of them are there?
2. Either draw a picture of the affine variety X, or give a precise verbal description of its shape.

(10 credits)

Answer:

1. There are 16 points with integer coordinates on X – $(0, \pm1, 0)$, $(0, \pm2, \pm1)$, $(0, \pm3, 0)$, $(\pm1, 0, 0)$, $(\pm2, 0, \pm1)$ and $(\pm3, 0, 0)$.

2. The affine variety X is a torus lying in the x-y-plane centered around the origin, with radii $R = 2$ and $r = 1$. (The given implicit representation of this torus is maybe the simplest possible.)

Question 10: Find out which of the following 20 assertions are true and which are false (only true/false answers – correct answer: 1 credit, no answer: 0 credits, wrong or unclear answer: -1 credit, $\geqslant 0$ credits in total; answers must be marked by an 'X' in the box after either 'true' or 'false'):

1. Multivariate polynomial rings over a field are commutative.
   true (X) false (   )

2. Multivariate polynomial rings are principal ideal domains.
   true (   ) false (X)
   Otherwise all the Groebner base stuff would not be needed at all.

3. Multivariate polynomial rings in finitely many variables over a field satisfy the Ascending Chain Condition (ACC) for ideals.
   true (X) false (   )
   See Thm. 2.5.7, Page 76.

4. Multivariate polynomial rings in finitely many variables over a field satisfy the Descending Chain Condition (DCC) for ideals.
   true (   ) false (X)
   E.g. $\langle x \rangle \supset \langle x^2 \rangle \supset \langle x^3 \rangle \supset \ldots$ is an infinite descending chain of ideals.

5. For univariate polynomial rings, lex order, grlex order and grevlex order are the same.
   true (X) false (   )
   You don't know these terms from school where you have dealt with univariate polynomials, do you?

6. If K is a finite field and $n \in \mathbb{N}$, then every affine variety in K^n has only finitely many points.
   true (X) false (   )
   Note that K^n itself is finite.

7. Let K be a field and $n \in \mathbb{N}$. Then an affine variety in K^n has only finitely many points if and only if the polynomial ring in n variables over K is finite.
   true (   ) false (X)
   No polynomial ring is finite, but there are finite affine varieties anyway.

8. Let p be a prime, and let K be a field of characteristic p. Then every affine variety in K^n ($n \in \mathbb{N}$) is finite, and the number of its points is a power of p.
   true (   ) false (X)
   K^n itself is an affine variety, and if K is infinite, this affine variety is also infinite.

9. The affine varieties in $\mathbb{R}^1$ are precisely the finite subsets and $\mathbb{R}^1$ itself.
   true (X) false (   )
   A subset of $\mathbb{R}$ is the set of roots of a nonzero univariate polynomial if and only if it is finite.

10. The affine varieties in $\mathbb{R}^2$ are precisely the finite subsets and $\mathbb{R}^2$ itself.
true (   ) false (X)
Counterexample: $V(x^2 + y^2 - 1)$ is the unit circle around the origin, which has infinitely many points.

11. The affine varieties in $\mathbb{C}^1$ are precisely the finite subsets and $\mathbb{C}^1$ itself.
true (X) false (   )
See the explanation for $\mathbb{R}^1$ above.

12. The affine variety $V(x^2 + y^2 + 1) < \mathbb{R}^2$ is empty.
true (X) false (   )
Squares of real numbers are nonnegative.

13. The affine variety $V(x^2 + y^2 + 1) < \mathbb{F}_2^2$ is empty.
true (   ) false (X)
It is $(0, 1) \in V(x^2 + y^2 + 1)$.

14. An affine variety in $\mathbb{Q}^n$ ($n \in \mathbb{N}$) is either finite or countable.
true (X) false (   )
Note that $\mathbb{Q}^n$ itself is countable.

15. There is an algorithm which can always decide whether a given affine variety in $\mathbb{Q}^n$ ($n \in \mathbb{N}$) is empty or not.
true (   ) false (X)
This is the algorithmic unsolvability of Diophantine equations.

16. A subset $V \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$) is an affine variety if and only if it is convex.
true (   ) false (X)
Counterexample: the unit sphere in $\mathbb{R}^n$ is an affine variety which is not convex.

17. The set-theoretic union of finitely many affine varieties is also an affine variety.
true (X) false (   )
See Lemma 1.2.2 on Page 11.

18. The set-theoretic union of an arbitrary (possibly infinite) collection of affine varieties is also an affine variety.
true (   ) false (X)
If it would, every set of points would be an affine variety, since 1-point sets are.

19. Knowing a Groebner basis for an ideal $I < K[x_1, \ldots, x_n]$, the ideal membership problem for I is algorithmically decidable.
true (X) false (   )
If not, the notion of a Groebner basis would not make much sense.

20. The algorithm for multivariate polynomial division terminates always.
true (X) false (   )
Otherwise it would not be an algorithm.

(20 credits)