Final Test in MAT 641: Computational Algebra I

Stefan Kohl

Date and time: Tuesday, February 9, 2010, 17:00 - 19:00.
Name:

You have **120 minutes** of time to answer the 10 questions below. Write your answers to Questions 1 – 9 to the blank sheets of paper supplied to you, and answer Question 10 on this sheet. You are not allowed to use anything else than a pen.

Question 1: Give the definition of an *ideal* of a ring. (2 credits)

Question 2: State

1. the *Ideal Membership Problem* for ideals of multivariate polynomial rings and
2. the *Implizitization Problem* for affine varieties.

(3 credits – 1 for a. and 2 for b.)

Question 3: Give the definition of a *Groebner basis* of an ideal of a multivariate polynomial ring. (3 credits)

Question 4: State the *Hilbert Basis Theorem*. (2 credits)

Question 5: State the *Ascending Chain Condition* (ACC) and the *Descending Chain Condition* (DCC) for ideals of a ring. (2 credits)

Question 6: Give a polynomial $P \in \mathbb{R}[x, y, z]$ such that the affine variety $V(P) \subset \mathbb{R}^3$ is the unit sphere about the origin. (2 credits)

Question 7: Let $P := 4x^3yz^2 + xyz - 7y^6z + 8z^8 + 2xy - 1 \in \mathbb{R}[x, y, z]$. Write down the polynomial P

1. in lex order,
2. in grlex order and
3. in grevlex order.

Always assume $x > y > z$. In all cases, determine the multidegree, the leading coefficient, the leading monomial and the leading term of P with respect to the respective monomial ordering. (6 credits)

Question 8: Let $P := 4x^3y - 3xy^2z$, $a := x^2 - y^2z$, $b := xy + z^3$ and $c := y^2 - yz$.

1. Divide P by a, b and c (in this order), using lex order.
2. Find out whether or not $\{a, b, c\}$ is a Groebner basis for the ideal $\langle a, b, c\rangle$ (proof required). *Hint:* Try to divide by a, b and c in different order.

(10 credits)

Question 9: Let $X := V((x^2 + y^2 + z^2 + 3)^2 - 16(x^2 + y^2)) \subset \mathbb{R}^3$.

1. Determine the points with integer coordinates on the affine variety X. – How many of them are there?
2. Either draw a picture of the affine variety X, or give a precise verbal description of its shape.

(10 credits)

Question 10: Find out which of the following 20 assertions are true and which are false (only true/false answers – correct answer: 1 credit, no answer: 0 credits, wrong or unclear answer: -1 credit, $\geqslant 0$ credits in total; answers must be marked by an 'X' in the box after either 'true' or 'false'):

1. Multivariate polynomial rings over a field are commutative.
   true ( ) false ( )

2. Multivariate polynomial rings are principal ideal domains.
true ( ) false ( )

3. Multivariate polynomial rings in finitely many variables over a field satisfy the Ascending Chain Condition (ACC) for ideals.
true ( ) false ( )

4. Multivariate polynomial rings in finitely many variables over a field satisfy the Descending Chain Condition (DCC) for ideals.
true ( ) false ( )

5. For univariate polynomial rings, lex order, grlex order and grevlex order are the same.
true ( ) false ( )

6. If K is a finite field and $n \in \mathbb{N}$, then every affine variety in K^n has only finitely many points.
true ( ) false ( )

7. Let K be a field and $n \in \mathbb{N}$. Then an affine variety in K^n has only finitely many points if and only if the polynomial ring in n variables over K is finite.
true ( ) false ( )

8. Let p be a prime, and let K be a field of characteristic p. Then every affine variety in K^n ($n \in \mathbb{N}$) is finite, and the number of its points is a power of p.
true ( ) false ( )

9. The affine varieties in $\mathbb{R}^1$ are precisely the finite subsets and $\mathbb{R}^1$ itself.
true ( ) false ( )

10. The affine varieties in $\mathbb{R}^2$ are precisely the finite subsets and $\mathbb{R}^2$ itself.
true ( ) false ( )

11. The affine varieties in $\mathbb{C}^1$ are precisely the finite subsets and $\mathbb{C}^1$ itself.
true ( ) false ( )

12. The affine variety $V(x^2 + y^2 + 1) < \mathbb{R}^2$ is empty.
true ( ) false ( )

13. The affine variety $V(x^2 + y^2 + 1) < \mathbb{F}_2^2$ is empty.
true ( ) false ( )

14. An affine variety in $\mathbb{Q}^n$ ($n \in \mathbb{N}$) is either finite or countable.
true ( ) false ( )

15. There is an algorithm which can always decide whether a given affine variety in $\mathbb{Q}^n$ ($n \in \mathbb{N}$) is empty or not.
true ( ) false ( )

16. A subset $V \subseteq \mathbb{R}^n$ ($n \in \mathbb{N}$) is an affine variety if and only if it is convex.
true ( ) false ( )

17. The set-theoretic union of finitely many affine varieties is also an affine variety.
true ( ) false ( )

18. The set-theoretic union of an arbitrary (possibly infinite) collection of affine varieties is also an affine variety.
true ( ) false ( )

19. Knowing a Groebner basis for an ideal $I < K[x_1, \ldots, x_n]$, the ideal membership problem for I is algorithmically decidable.
true ( ) false ( )

20. The algorithm for multivariate polynomial division terminates always.
true ( ) false ( )

(20 credits)

– Good luck!

Maximum possible number of credits: 60.
Grade = (number of credits)/6, rounded to the nearest integer.