Midterm Test in MAT 641: Computational Algebra I
Answers to the Test Questions

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Question 1: Give the definition of a *field*. (3 credits)

Answer: There are various ways to write down the definition of a field. – For example:

- A *field* is a commutative ring with one, in which every nonzero element is a unit.
- A *field* is an abelian group $(K, +)$ endowed with a second binary operation $\cdot$ such that
  - $(K \setminus \{0\}, \cdot)$ is an abelian group, and
  - for all $a, b, c \in K$ it is $a \cdot (b + c) = a \cdot b + a \cdot c$.
- A *field* is a triple $(K, +, \cdot)$ consisting of a set K and binary operations $+, \cdot : K \times K \to K$ such that the following hold:
  1. $\exists 0 \in K : \ \forall a \in K \ a + 0 = a$ (existence of neutral element with respect to addition).
  2. $\forall a, b, c \in K \ a + (b + c) = (a + b) + c$ (associativity of addition).
  3. $\forall a, b \in K \ a + b = b + a$ (commutativity of addition).
  4. $\forall a \in K \ \exists -a \in K : \ a + -a = 0$ (existence of additive inverses).
  5. $\exists 1 \in K : \ \forall a \in K \ a \cdot 1 = a$ (existence of neutral element with respect to multiplication).
  6. $\forall a, b, c \in K \ a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity of multiplication).
  7. $\forall a, b \in K \ a \cdot b = b \cdot a$ (commutativity of multiplication).
  8. $\forall a \in K \ \exists a^{-1} \in K : \ a \cdot a^{-1} = 1$ (existence of multiplicative inverses).
  9. $\forall a, b, c \in K \ a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity).

All of these – and many other variations – were accepted.

Question 2: Give the definition of the *characteristic* of a field. (2 credits)

Answer: The *characteristic* of a field K is the least positive number of ones one needs to add in K in order to get zero in case such number exists, and zero otherwise.

Question 3: Precisely for which positive integers n is there a finite field with n elements? (1 credit)

Answer: There is a finite field with n elements if and only if n is a prime power.

Question 4: Why is it not necessary to distinguish between left and right ideals in polynomial rings? (2 credits)

Answer: Because polynomial rings are commutative.

Question 5: When exactly is the polynomial ring $K[x_1, \ldots, x_n]$ in n variables over a field K a principal ideal domain? (2 credits)

Answer: If and only if $n = 1$.

Question 6: Give an example of a polynomial $f \in \mathbb{F}_3[x] \setminus \{0\}$ such that $f : \mathbb{F}_3 \to \mathbb{F}_3$ is the zero function. (2 credits)

Answer: $x^3 + 2x$ (for example).

Question 7: Give the definition of an *affine variety.* (2 credits)

Answer: An *affine variety* is the set of common zeros of a finite set of polynomials in finitely many variables over a field.

Question 8: Give the definition of a *rational parametric representation* of an affine variety. (4 credits)

Answer: A *rational parametric representation* of an affine variety $V = \langle f_1 \ldots, f_s \rangle \subset k^n$ is a set of rational functions $r_1, \ldots, r_n \in k(t_1, \ldots, t_m)$ such that all points given by $x_i = r_i(t_1, \ldots, t_m)$, $i \in \{1, \ldots, n\}, t_j \in k$ lie in V, and such that V is the smallest affine variety containing these points.

Question 9: When is an affine variety called *unirational*? (2 credits)

Answer: If and only if it has a rational parametric representation.

Question 10: Describe the location of a *Bezier cubic* relative to its control polygon. (2 credits)

Answer: It lies inside.

Question 11: Is the union of infinitely many affine varieties necessarily also an affine variety? – Either prove or disprove. (5 credits)

Answer: Every point $p \in \mathbb{R}$ is an affine variety, since there is always a polynomial $f \in \mathbb{R}[x]$ which has p as its only zero. While infinite proper subsets of $\mathbb{R}$ are unions of 1-point subsets, they are not affine varieties, since the number of zeros of a nonzero polynomial in $\mathbb{R}[x]$ is bounded above by its degree. So the answer to the question is "no".

Question 12: Let V be the affine variety $\langle x^4 + y^4 - 1 \rangle \subset \mathbb{Q}^2$. Determine $|V|$. (No proof required!) (3 credits)

Answer: It is $V = \{(0, -1), (0, 1), (-1, 0), (1, 0)\}$, and so $|V| = 4$ – cf. Exercise 2.11 in the book.