Final Test in MAT 642: Computational Algebra II
Answers

Stefan Kohl

June 23, 2010

Question 1: Give the definitions of a

- *basis*, of a
- *Groebner basis*, of a
- *minimal Groebner basis* and of a
- *reduced Groebner basis*

of an ideal of a polynomial ring, and explain the importance of these four terms. (8 parts, worth 2 credits each – 16 credits in total)

Answer:

- A *basis* $B \subset I$ of an ideal $I \subset K[x_1, \ldots, x_n]$ is a subset such that I consists of all linear combinations of the elements of B.

  Since every ideal of a polynomial ring in finitely many variables is finitely generated, bases allow to represent such ideals in a finite amount of memory.

- A basis of an ideal of a polynomial ring is called a *Groebner basis* if the ideal generated by the leading terms of the basis elements contains the leading terms of all elements.

  Groebner bases allow to solve the ideal membership problem, since dividing a polynomial by a Groebner basis yields remainder 0 iff the polynomial is an element of the corresponding ideal.

- A Groebner basis is called a *minimal Groebner basis* if its elements are monic and if the leading term of none of its elements lies in the ideal generated by the leading terms of the others.

  Minimal Groebner bases do not contain superfluous elements and thus their use avoids wasting memory for the storage of such elements.

- A minimal Groebner basis is called a *reduced Groebner basis* if no monomial of any of its elements lies in the ideal generated by the leading terms of the other elements.

  Since the reduced Groebner basis of an ideal is unique, reduced Groebner bases can be used to decide equality of ideals.

Question 2: State the *Buchberger algorithm.* (6 credits)

Answer: In its simplest form, the *Buchberger algorithm* augments a given basis of a polynomial ideal by successively adding the remainders of S-polynomials of pairs of basis elements under division by the basis computed so far. It terminates once all such remainders are zero.

Question 3: Explain how to solve a system of polynomial equations by computer. Is it always a good idea to proceed in the same way in hand computations? – Explain. (6 credits)

Answer: By computer, one first computes a Groebner basis for lex order. Then, one solves the equation which involves the smallest number of variables. Finally, one extends the partial solutions obtained in this way to complete solutions by insertion into the remaining equations.

By hand, computing a Groebner basis for lex order can be very tedious. On the other hand, in easy examples one often sees ways to find the solutions much quicker. "Non-easy" cases are not doable by hand, anyway.

Question 4: Find all solutions $(x, y, z) \in \mathbb{C}^3$ of the following system of polynomial equations:

$$\begin{aligned} x^2y + 2yz - z^3 &= 1 \\ x^3 - 2x^2y + 2z^3 &= 2 \\ \tfrac{1}{4}x^3 + yz &= 3 \end{aligned}$$

(6 credits)

Answer: Two times the first equation plus the second equation minus four times the third equation yields $0 = -8$, which is false. Hence there are no solutions.

Question 5: Find all solutions $(x, y, z) \in \mathbb{C}^3$ of the following system of polynomial equations:

$$\begin{aligned} x + y + z &= 0 \\ x^2y + 2xy^2 + xyz + y^3 + y^2z &= 1 \end{aligned}$$

(6 credits)

Answer: If $x + y + z$ is zero, then $x^2y + 2xy^2 + xyz + y^3 + y^2z = (x + y + z) \cdot (xy + y^2)$ is zero as well, and not one. Hence there are no solutions.

Question 6: Try to determine all solutions $(x, y) \in \mathbb{C}^2$ of the following system of polynomial equations:

$$\begin{aligned} x^4 - y &= 1 \\ x - y^4 &= 1 \end{aligned}$$

Which problem do you encounter when trying to write down the solutions? (6 credits)

Answer: Obviously, one solution is $(x, y) = (1, 0)$. By the first equation, we have $y = x^4 - 1$. Inserting this into the second equation yields $x^{16} - 4x^{12} + 6x^8 - 4x^4 - x + 2 = 0$. Due to the obvious solution, one root must be 1, and indeed this polynomial factors as $(x-1)\cdot(x^{15}+x^{14}+x^{13}+x^{12}-3x^{11}-3x^{10}-3x^9-3x^8+3x^7+3x^6+3x^5+3x^4-x^3-x^2-x-2)$. By hand, one gets stuck here – and observing this was enough to get all the credits.

With GAP one can check easily that the factor of degree 15 is irreducible, and one finds that its Galois group is S_{15}. Therefore the solutions other than $(x, y) = (1, 0)$ cannot be expressed in terms of radicals over $\mathbb{Q}$.

Question 7: Compute the reduced Groebner bases for the ideal $\langle x^2 + y - 1, xy + y - 1 \rangle \subset \mathbb{C}[x, y]$ for

1. lex order and

2. grlex order,

where $x > y$. (6 credits)

Answer: The reduced Groebner basis for lex order is $\{x + y^2 - 1, y^3 - 2y + 1\}$, and the one for grlex order is $\{x^2 + y - 1, xy + y - 1, y^2 + x - 1\}$.

Question 8: Is there a constant c such that all ideals of $\mathbb{C}[x, y]$ can be generated by c or fewer polynomials? – Either prove or disprove. (8 credits)

Answer: The assertion is false, since for any $n \in \mathbb{N}$ the ideal $\langle xy^n, x^2y^{n-1}, x^3y^{n-2}, \ldots, x^ny \rangle$ cannot be generated by fewer than n polynomials. To see this, draw the corresponding graphs.