Midterm 1 in MAT 642: Computational Algebra II
Answers to the Test Questions

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Question 1: Give the definition of a *Groebner basis* of an ideal of a multivariate polynomial ring over a field. (3 credits)

Answer: A basis of such an ideal is called a *Groebner basis* (w.r.t. a certain monomial order) if the ideal generated by the leading terms of the basis elements contains the leading terms of all elements of the ideal.

Question 2: Explain the importance of Groebner bases in the context of the membership problem for ideals of multivariate polynomial rings. (3 credits)

Answer: Multivariate polynomial division by the elements of a Groebner basis can decide the ideal membership problem. In contrast, when dividing a polynomial by the elements of a basis other than a Groebner basis, one may get a nonzero remainder also if the polynomial is contained in the ideal.

Question 3: State the *Hilbert Basis Theorem.* (2 credits)

Answer: Ideals of polynomial rings in finitely many variables over a field are finitely generated.

Question 4: Give an example of a polynomial $f \in \mathbb{R}[x, y]$ whose monomials are ordered in pairwise different ways in lex order, grlex order and grevlex order. Assume that $x > y$. How many monomials does such a polynomial f need to have at least? (3 credits)

Answer: An example with 3 monomials is $x^2 + xy^2 + y^3$ (lex order), which reads $xy^2 + y^3 + x^2$ in grlex order and $y^3 + xy^2 + x^2$ in grevlex order. Since 2 monomials cannot be permuted in 3 different ways, 3 is the least possible number of monomials of such a polynomial.

Question 5: Let $I := \langle x^2 + y, x + y^2 \rangle \subset \mathbb{R}[x, y]$ be an ideal. Assume lex order with $x > y$. Either prove or disprove that $\{x^2 + y, x + y^2\}$ is a Groebner basis for I. (4 credits)

Answer: The leading term of $(x^2 + y) + (-x + y^2)(x + y^2) = y^4 + y \in I$ is y^4, which is not contained in the ideal generated by the leading terms x^2 and x of the given basis elements. Therefore the given basis is *not* a Groebner basis.

Question 6: Either prove or disprove the assertion that every ideal of $\mathbb{R}[x, y]$ has a generating set with not more than 2 elements. (3 credits)

Answer: For example the ideal $\langle xy^3, x^2y^2, x^3y \rangle \subset \mathbb{R}[x, y]$ has no generating set with 2 or fewer elements (draw the graph!).

Question 7: Determine the number of points with integer coordinates on the affine variety

$$V(x^3 + xz^2 - y^2 + xyz) \;\subset\; \mathbb{R}^3$$

(proof required!). (4 credits)

Answer: For $z = 0$ we obtain the equation $x^3 = y^2$, which is satisfied by $x = n^2$ and $y = n^3$ for any integer n. Hence there are infinitely many points with integer coordinates on the given affine variety.

Question 8 (answered): Here are the 8 varieties together with their descriptions:

1. $V(x^2 + y^2 + z^2 - 1, (x^2 - 1)(y^2 - 1)(z^2 - 1))$.
   Description: h.) the set $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$.
2. $V(x^2 + y^2 - z^2, x)$.
   Description: c.) two intersecting lines.
3. $V((x^2 - 1)(y^2 - 1)(z^2 - 1), xyz)$.
   Description: a.) 12 lines.
4. $V(x^2 + y^2 + z^2 - 1, (x - 1)^2 + (y - 2)^2 + (z - 3)^2 - 1)$.
   Description: b.) the empty set.
5. $V(x^2 + y^2 - z^2, x^2 + y^2 + z^2 - 1)$.
   Description: f.) two circles.
6. $V((x^2 + y^2 + z^2 + 24)^2 - 100(x^2 + y^2))$.
   Description: g.) a torus.
7. $V(x^2 + 2y^2 + 3z^2 - 4)$.
   Description: e.) an ellipsoid.
8. $V(x^2 + y^2 - z^2, z)$.
   Description: d.) the set $\{(0, 0, 0)\}$.

(8 credits – 1 for each correct attribution)