Midterm 1 in MAT 642: Computational Algebra II

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Date and time: Tuesday, April 6, 2010, 14:00 - 15:00.
Room: C 411

NAME:

You have **60 minutes** of time to answer the 8 questions below. You are not allowed to use anything else than a pen and blank sheets of paper.

Question 1: Give the definition of a *Groebner basis* of an ideal of a multivariate polynomial ring over a field. (3 credits)

Question 2: Explain the importance of Groebner bases in the context of the membership problem for ideals of multivariate polynomial rings. (3 credits)

Question 3: State the *Hilbert Basis Theorem*. (2 credits)

Question 4: Give an example of a polynomial $f \in \mathbb{R}[x, y]$ whose monomials are ordered in pairwise different ways in lex order, grlex order and grevlex order. Assume that $x > y$. How many monomials does such a polynomial f need to have at least? (3 credits)

Question 5: Let $I := \langle x^2 + y, x + y^2 \rangle \subset \mathbb{R}[x, y]$ be an ideal. Assume lex order with $x > y$. Either prove or disprove that $\{x^2 + y, x + y^2\}$ is a Groebner basis for I. (4 credits)

Question 6: Either prove or disprove the assertion that every ideal of $\mathbb{R}[x, y]$ has a generating set with not more than 2 elements. (3 credits)

Question 7: Determine the number of points with integer coordinates on the affine variety

$$V(x^3 + xz^2 - y^2 + xyz) \subset \mathbb{R}^3$$

(proof required!). (4 credits)

Question 8: Find out which description fits to which affine variety in $\mathbb{R}^3$. – Here are the varieties:

1. $V(x^2 + y^2 + z^2 - 1, (x^2 - 1)(y^2 - 1)(z^2 - 1))$.

   Description:

2. $V(x^2 + y^2 - z^2, x)$.

   Description:

3. $V((x^2 - 1)(y^2 - 1)(z^2 - 1), xyz)$.

   Description:

4. $V(x^2 + y^2 + z^2 - 1, (x - 1)^2 + (y - 2)^2 + (z - 3)^2 - 1)$.

   Description:

5. $V(x^2 + y^2 - z^2, x^2 + y^2 + z^2 - 1)$.

   Description:

6. $V((x^2 + y^2 + z^2 + 24)^2 - 100(x^2 + y^2))$.

   Description:

7. $V(x^2 + 2y^2 + 3z^2 - 4)$.

   Description:

8. $V(x^2 + y^2 - z^2, z)$.

   Description:

... and here are the descriptions: a.) 12 lines, b.) the empty set, c.) two intersecting lines, d.) the set $\{(0, 0, 0)\}$, e.) an ellipsoid, f.) two circles, g.) a torus, h.) the set $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$.
(8 credits – 1 for each correct attribution)

– Good luck!

Maximum possible number of credits: 30.
Grade = (number of credits)/3, rounded to the nearest integer.