Midterm 2 in MAT 642: Computational Algebra II
Answers

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Question 1: When is a Groebner basis said to be *minimal*, and when is it said to be *reduced*? (4 credits)

Answer: A Groebner basis G for a polynomial ideal I is said to be *minimal* if

1. $\mathrm{LC}(p) = 1$ for all $p \in G$ and
2. $\forall p \in G \;\; \mathrm{LT}(p) \notin \langle \mathrm{LT}(G \setminus \{p\}) \rangle$.

It is said to be *reduced* if in addition for all $p \in G$ no monomial of p lies in $\langle \mathrm{LT}(G \setminus \{p\}) \rangle$.

Question 2: Explain the importance of the concept of a *reduced* Groebner basis. (3 credits)

Answer: The reduced Groebner basis of a polynomial ideal with respect to a given monomial order is unique. Therefore reduced Groebner bases can be used to check polynomial ideals for equality.

Question 3: Let $n \in \mathbb{N}$, let $r \in \{0, \dots, n\}$ and let K be some field. Give the definition of the r-th *elimination ideal* I_r of an ideal I of $K[x_1, \dots, x_n]$. (3 credits)

Answer: $I_r = I \cap K[x_{r+1}, \dots, x_n]$.

Question 4: State the *Elimination Theorem.* (4 credits)

Answer: Let $I \subset K[x_1, \dots, x_n]$ be an ideal and let G be a Groebner basis for I with respect to lex order where $x_1 > x_2 > \cdots > x_n$. Then for every $0 \leqslant l \leqslant n$, the set $G_l = G \cap K[x_{l+1}, \dots, x_n]$ is a Groebner basis of the l-th elimination ideal I_l.

Question 5: Briefly explain why the *Buchberger Algorithm* terminates. (4 credits)

Answer: Let $B = B_0$ be the initial basis with which the Buchberger Algorithm is invoked, and let $B_1, B_2, \dots$ be the augmented bases in the order in which they are encountered in the passes through the main loop of the algorithm. Then, $\langle \mathrm{LT}(B_0) \rangle, \langle \mathrm{LT}(B_1) \rangle, \dots$ is an ascending chain of ideals. Since polynomial rings in finitely many variables satisfy the ACC for ideals, this chain eventually gets constant. Once this happens, the Buchberger Algorithm terminates.

Question 6: Compute reduced Groebner bases for the following ideals for lexicographic order:

1. $\langle a, b, c, d, e, f \rangle \subset \mathbb{C}[a, b, c, d, e, f]$.
2. $\langle x^2y^3z^2 + \sqrt{2}xy^2 - 5x^2y + 7, 8 + \sqrt{5}\pi^2, x^2y^2z - 3x^3y + 2xyz^2 \rangle \subset \mathbb{C}[x, y, z]$.

(4 credits)

Answer: $\{a, b, c, d, e, f\}$ respectively $\{1\}$. – Note that one generator of the second ideal is a constant, which means that this ideal is the unit ideal.

Question 7: Solve the system of equations

$$\begin{aligned} x^2 + y^2 &= 1, \\ xy + x + y &= 1 \end{aligned}$$

for $x, y \in \mathbb{C}$. – All solutions need to be found. (8 credits – 2 for the trivial and 6 for the nontrivial solutions)

Answer: The reduced Groebner basis for the ideal $\langle x^2 + y^2 - 1, xy + x + y - 1\rangle$ with respect to lex order is

$$\{x + \tfrac{1}{2}y^3 + \tfrac{1}{2}y^2 - 1, y^4 + 2y^3 + y^2 - 4y\}.$$

The second generator factors over $\mathbb{Q}$ as $y(y-1)(y^2+3y+4)$, therefore its roots are 0, 1 and $-\frac{3}{2} \pm \frac{1}{2}\sqrt{7}i$. Inserting these values of y into the first generator – or just using the symmetry of the system of equations with respect to x and y – yields the 4 solutions

$$(x, y) \in \{(0, 1), (1, 0), (-\tfrac{3}{2} - \tfrac{1}{2}\sqrt{7}i, -\tfrac{3}{2} + \tfrac{1}{2}\sqrt{7}i), (-\tfrac{3}{2} + \tfrac{1}{2}\sqrt{7}i, -\tfrac{3}{2} - \tfrac{1}{2}\sqrt{7}i)\}.$$